Invariants de type fini des variétés de dimension trois et structures spinorielles

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Titre

INVARIANTS DE TYPE FINI DES VARIÉTÉS DE DIMENSION TROIS ET STRUCTURES SPINORIELLES

Jury

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Introduction

S'inspirant de l'approche de V. Vassiliev pour l'étude des noeuds, T. Ohtsuki a introduit dans [**O**] une *théorie d'invariants de type fini* des sphères d'homologie entière. Puis, T. Cochran et P. Melvin l'ont généralisée dans [**CM**] aux 3-variétés compactes orientées.
De façon indépendante, M. Goussarov et K. Habiro ont introduit une autre théorie d'invariants de type fini pour les 3-variétés compactes orientées et leurs entrelacs. Développée dans [**Go**], [**GGP**] et [**Hr**], leur théorie s'accompagne d'une technique calculatoire efficace et originale, appelée *calcul de clovers* ou *calcul de claspers*, et définie par des opérations du type "couper/recoller".
La restriction aux sphères d'homologie entière de la théorie de Goussarov-Habiro se trouve être équivalente à celle d'Ohtsuki, et est assez bien comprise (au moins pour les coefficients rationnels). En revanche, pour les 3-variétés homologiquement non-triviales, la théorie de Goussarov-Habiro ne coïncide pas avec celle de Cochran-Melvin et reste encore méconnue.

Une 3-variété compacte orientée peut être équipée de structures supplémentaires, appelées *Spin*-structures et $Spin^c$-structures : lorsque la variété est homologiquement non-triviale, elle peut être munie de plusieurs de ces structures. Depuis les travaux de V. Rochlin, l'émergence de *Spin*-structures puis de $Spin^c$-structures, dans les thèmes de la topologie de basse dimension s'est répétée. Par exemple, les *Spin*-structures se sont avérées très utiles pour l'étude des mapping class groups des surfaces, autant que pour celle des invariants quantiques des 3-variétés compactes orientées. De même, les travaux de V. Turaev ont révélé l'importance des $Spin^c$-structures dans la théorie des torsions combinatoires.

Dans cette thèse, nous proposons[1], et nous étudions, des raffinements de la théorie de Goussarov-Habiro aux 3-variétés compactes orientées munies de *Spin*-structures ou de $Spin^c$-structures. Plus généralement, nous nous efforçons d'illustrer l'intérêt que présente la prise en compte de ces structures pour l'étude de la théorie de Goussarov-Habiro.

Cette thèse est constituée de cinq chapitres que nous introduisons maintenant successivement.

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Le premier chapitre expose, de façon unifiée et détaillée, des raffinements *Spin* et $Spin^c$ de la théorie d'invariants de type fini de Goussarov-Habiro.

A cet effet, nous débutons par quelques rappels sur la théorie "brute". Puis, nous révisons les *Spin*-structures et les $Spin^c$-structures, et leurs équivalents tels que les parallélisations et les structures d'Euler géométriques. Nous mettons alors sur pied des techniques de recollement des structures spinorielles (complexes). Nous

[1] Goussarov et Habiro ont annoncé dans [**Go**] et [**Hr**] la possibilité de raffiner leur théorie aux 3-variétés munies de *Spin*-structures, tandis que Cochran et Melvin ont adapté leur théorie au cas *Spin*.

montrons aussi, pour toute 3-variété M compacte orientée à bord et pour toute *Spin*-structure σ sur ∂M, l'existence de *$Spin^c$-structures sur M relatives à σ*. Noté $Spin^c(M,\sigma)$, l'espace de ces structures joue un rôle essentiel dans la suite.

Ces techniques de recollement nous permettent de donner une signification *Spin* et $Spin^c$ à ces mouvements chirurgicaux qui consistent à modifier une 3-variété compacte orientée en la twistant le long d'une surface fermée connexe scindante par un difféomorphisme agissant trivialement en homologie. Nous pouvons alors, comme annoncé, raffiner la théorie de Goussarov-Habiro aux 3-variétés compactes orientées munies de *Spin*-structures ou de $Spin^c$-structures. Le calcul de clovers s'étend aux cas *Spin* et $Spin^c$.

...

Le deuxième chapitre reprend la publication [**Ms**] et s'intéresse au raffinement *Spin* de la théorie de Goussarov-Habiro.

La première question soulevée par une théorie d'invariants de type fini est la caractérisation de ses invariants de degré 0. Pour la théorie de Goussarov-Habiro et dans le cas des variétés fermées orientées, un théorème de S. Matveev exprime que les invariants de degré 0 coïncident avec les invariants du premier nombre de Betti de M et de la classe d'isomorphisme de la forme d'enlacement de M : $TH_1(M)\otimes TH_1(M) \xrightarrow{\lambda_M} \mathbf{Q}/\mathbf{Z}$, où $TH_1(M)$ désigne le sous-groupe de torsion du premier groupe d'homologie à coefficients entiers $H_1(M)$.
Maintenant, si (M,σ) est une 3-variété fermée orientée munie d'une *Spin*-structure, on sait lui associer une forme quadratique $TH_1(M) \xrightarrow{\phi_{M,\sigma}} \mathbf{Q}/\mathbf{Z}$ au-dessus de λ_M, ainsi que son invariant de Rochlin $R(M,\sigma)\in \mathbf{Z}_{16}$. Nous montrons le raffinement suivant du théorème de Matveev.

THÉORÈME 1. *Soient (M,σ) et (M',σ') des Spin-variétés fermées de dimension 3. Les assertions suivantes sont alors équivalentes :*

(1) *(M,σ) et (M',σ') ne sont pas distinguées par les invariants de degré 0 ;*

(2) *il existe un isomorphisme $H_1(M) \xrightarrow{\psi} H_1(M')$ tel que $\phi_{M,\sigma}=\phi_{M',\sigma'}\circ\psi|$;*

(3) *il existe un isomorphisme $H_1(M) \xrightarrow{\psi} H_1(M')$ tel que $\lambda_M=\lambda_{M'}\circ(\psi|)^{\otimes 2}$, et $R(M,\sigma)=R(M',\sigma')$ mod 8.*

L'équivalence entre les assertions (2) et (3) du Théorème 1 est obtenue comme un corollaire du résultat algébrique suivant.

THÉORÈME 2. *Soient $G \xrightarrow{q} \mathbf{Q}/\mathbf{Z}$ et $G' \xrightarrow{q'} \mathbf{Q}/\mathbf{Z}$ des formes quadratiques non-dégénérées sur des groupes abéliens finis. Les assertions suivantes sont alors équivalentes :*

(1) *il existe un isomorphisme $G \xrightarrow{\psi} G'$ tel que $q=q'\circ\psi$;*

(2) *il existe un isomorphisme $G \xrightarrow{\psi} G'$ tel que $b_q=b_{q'}\circ\psi^{\otimes 2}$ et $\gamma(q)=\gamma(q')\in\mathbf{C}$.*

Ici, b_q est la forme bilinéaire symétrique associée à q, définie par $b_q(x,y)=q(x+y)-q(x)-q(y)$, et $\gamma(q)$ désigne la somme de Gauss de la forme quadratique q.

Nous démontrons aussi que l'invariant de Rochlin (non-réduit) est de degré 1, assurant ainsi la non-trivialité de la théorie raffinée par rapport à la théorie brute en degrés 0 et 1.

...

Le troisième chapitre reprend la publication [**MM**], **qui est un travail commun avec J.-B. Meilhan**. Nous y étudions les cylindres d'homologie au-dessus de surfaces compactes orientées.

Si Σ est une surface compacte orientée, un cylindre d'homologie au-dessus de Σ est un cobordisme d'homologie au-dessus de Σ avec une condition supplémentaire de trivialité homologique. Les cylindres d'homologies sont d'une grande importance aussi bien dans les travaux de Goussarov que dans ceux de Habiro. Pour l'un, les cylindres d'homologie doivent être des "objets modèles" dans la théorie d'invariants de type fini. Pour l'autre, les cylindres d'homologie sur la surface Σ doivent aider à la compréhension de son mapping class group ; ils forment en effet un monoïde dans lequel se plonge le groupe de Torelli de Σ.

Dans ce chapitre, nous nous intéressons au cas où Σ est une surface compacte orientée sans bord, ou avec une seule composante de bord. Grâce à des extensions aux cylindres d'homologie du premier homomorphisme de Johnson et des homomorphismes de Birman-Craggs (définis originellement sur le groupe de Torelli de Σ), nous prouvons :

Théorème 3. *Soit Σ une surface compacte orientée avec au plus 1 composante de bord et soient M, M' deux cylindres d'homologie au-dessus de Σ. Alors M et M' ne sont pas distingués par les invariants de type fini de degré 1 si, et seulement si, ils ne sont distingués ni par le premier homomorphisme de Johnson, ni par les homomorphismes de Birman-Craggs.*

Ce résultat est à rapprocher du théorème de Johnson sur la structure de l'abélianisation du groupe de Torelli de Σ.
La preuve utilise notamment le raffinement *Spin* de la théorie de Goussarov-Habiro et illustre bien en ce sens l'apport de la théorie raffinée à l'étude de la théorie brute (au moins pour les invariants de degré 1). En particulier, nous donnons une interprétation diagrammatique des homomorphismes de Birman-Craggs, basée sur un système de poids pour l'invariant de Rochlin.

...

Le quatrième chapitre est constitué de la prépublication [**DM**], **écrite en collaboration avec F. Deloup**. Celle-ci est consacrée à l'étude du rôle joué par les fonctions quadratiques dans la topologie des variétés fermées orientées de dimension 3, et s'intéresse ainsi au raffinement $Spin^c$ de la théorie de Goussarov-Habiro.

Nous y démontrons le résultat suivant.

Théorème 4. *Si M est une 3-variété orientée fermée, il existe alors un plongement affine :*

$$Spin^c(M) \hookrightarrow Quad(L_M),$$

associant à toute $Spin^c$-structure σ une fonction quadratique $\phi_{M,\sigma}$ au-dessus de la forme L_M.

Ici, L_M est la forme bilinéaire symétrique obtenue par composition de la forme d'enlacement λ_M avec $B^{\otimes 2}$, où $H_2(M;\mathbf{Q}/\mathbf{Z}) \xrightarrow{B} TH_1(M)$ est l'homomorphisme de Bockstein associé à la suite exacte courte de coefficients :

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 0.$$

La *fonction* quadratique $\phi_{M,\sigma}$ n'est pas nécessairement une *forme* quadratique (elle n'est pas nécessairement homogène). En revanche, si σ provient d'une *Spin*-structure *via* la flèche naturelle $Spin(M) \longrightarrow Spin^c(M)$, qui peut ne pas être injective ni surjective, elle coïncide avec la forme quadratique sus-mentionnée.

Ayant au préalable raffiné le théorème de Kirby aux $Spin^c$-variétés fermées de dimension 3, nous avons défini la fonction quadratique $\phi_{M,\sigma}$ à partir d'une présentation chirurgicale de (M,σ). Grâce aux formules de chirurgies pour la torsion de Reidemeister-Turaev, nous démontrons alors :

THÉORÈME 5. *Si* (M,σ) *est une sphère d'homologie rationnelle munie d'une structure spinorielle complexe, alors sa fonction quadratique* $\phi_{M,\sigma}$ *est déterminée par sa torsion abélienne maximale de Reidemeister-Turaev :* $\tau(M,\sigma) \in \mathbf{Q}\,[H_1(M)]$.

Nous donnons aussi une définition intrinsèque pour $\phi_{M,\sigma}$ (*i.e.* ne faisant pas référence à la dimension 4) lorsque σ est regardée comme une structure d'Euler géométrique.

Nous caractérisons enfin les invariants de degré 0 du raffinement $Spin^c$ de la théorie de Goussarov-Habiro en démontrant la généralisation suivante du théorème de Matveev.

THÉORÈME 6. *Soient* (M,σ) *et* (M',σ') *des* $Spin^c$*-variétés fermées de dimension* 3. *Les assertions suivantes sont alors équivalentes :*

(1) (M,σ) *et* (M',σ') *ne sont pas distinguées par les invariants de degré* 0 *;*

(2) *il existe un isomorphisme* $H_1(M) \xrightarrow{\psi} H_1(M')$ *tel que* $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \psi^\sharp$ *;*

(3) *il existe un isomorphisme* $H_1(M) \xrightarrow{\psi} H_1(M')$ *vérifiant :*
- $\lambda_M = \lambda_{M'} \circ (\psi|)^{\otimes 2}$,
- $\psi\left(P^{-1}c(\sigma)\right) = P^{-1}c(\sigma') \in H_1(M')$,
- *si* s *et* s' *sont des sections des homomorphismes de Bockstein* B *pour respectivement* M *et* M', *compatibles dans le sens où ce diagramme commute :*

$$\begin{tikzcd}
H_2\,(M;\mathbf{Q}/\mathbf{Z}) & TH_1(M) \arrow[l, "s"'] \arrow[d, "\simeq"', "\psi|"] \\
H_2\,(M';\mathbf{Q}/\mathbf{Z}) \arrow[u, "\psi^\sharp", "\simeq"'] & TH_1\,(M') \arrow[l, "s'"'],
\end{tikzcd}$$

alors : $\gamma\left(\phi_{M,\sigma} \circ s\right) = \gamma\left(\phi_{M',\sigma'} \circ s'\right) \in \mathbf{C}$.

Ici, $c(\sigma) \in H^2(M)$ est la classe de Chern de la $Spin^c$-structure σ et P désigne un isomorphisme de dualité de Poincaré. Enfin, $\psi^\sharp : H_2\,(M';\mathbf{Q}/\mathbf{Z}) \longrightarrow H_2\,(M;\mathbf{Q}/\mathbf{Z})$ est l'isomorphisme dual de ψ pour les formes d'intersection de M et M'.
L'équivalence entre les assertions (2) et (3) du Théorème 6 se prouve à partir de la généralisation suivante du Théorème 2, prouvée différemment.

THÉORÈME 7. *Soient* $G \xrightarrow{q} \mathbf{Q}/\mathbf{Z}$ *et* $G' \xrightarrow{q'} \mathbf{Q}/\mathbf{Z}$ *des fonctions quadratiques non-dégénérées sur des groupes abéliens finis. Les assertions suivantes sont alors équivalentes :*

(1) *il existe un isomorphisme* $G \xrightarrow{\psi} G'$ *tel que* $q = q' \circ \psi$ *;*

(2) *il existe un isomorphisme* $G \xrightarrow{\psi} G'$ *tel que* $b_q = b_{q'} \circ \psi^{\otimes 2}$, $d_q = d_{q'} \circ \psi$ *et :* $\gamma\,(q) = \gamma\,(q') \in \mathbf{C}$.

Ici, $d_q \in Hom\,(G,\mathbf{Q}/\mathbf{Z})$ est le défaut d'homogénéité de q défini par $d_q(x) = q(x) - q(-x)$ pour tout $x \in G$.

...

Dans le cinquième et dernier chapitre de cette thèse, nous nous intéressons aux torsions abéliennes de Reidemeister, telles qu'elles ont été raffinées par Turaev.

On rappelle, lors des premières pages de ce chapitre, comment associer à toute $Spin^c$-variété fermée (M,σ) et à tout homomorphisme d'anneaux $\mathbf{Z}\,[H_1(M)] \xrightarrow{\varphi} \mathbf{F}$ à valeurs dans un corps commutatif $\mathbf{F}$, la *torsion abélienne de Reidemeister-Turaev* :

$$\tau^\varphi\,(M,\sigma) \in \mathbf{F}.$$

Si la $Spin^c$-variété fermée (M', σ') s'obtient en twistant (M, σ) le long d'une surface fermée connexe scindante S par un élément h du groupe de Torelli de S (tel que cela aura été défini au Chap. 1), nous évaluons la variation de torsion. En d'autres termes, nous cherchons $C \in \mathbf{F}$ tel que $\tau^{\varphi'}(M', \sigma') = C \cdot \tau^{\varphi}(M, \sigma)$. Ce scalaire C n'est identifié qu'à un facteur multiplicatif près dans $\varphi(H_1(M))$. Il dépend notamment de la représentation de Magnus $M(h) \in GL_{2g}(\mathbf{Z})$ du difféomorphisme h. Sous certaines conditions sur le corps $\mathbf{F}$ et l'homomorphisme φ, le scalaire C est déterminé. Nous déduisons par exemple de cette étude que :

THÉORÈME 8. *Si h agit trivialement sur le deuxième quotient résoluble du groupe $\pi_1(S)$, nous avons alors :*

$$\tau^{\varphi'}(M', \sigma') = C \cdot \tau^{\varphi}(M, \sigma) \in \mathbf{F},$$

où $C \in \varphi(H_1(M))$ est une constante indépendante de σ, qui est égale à 1 sous certaines conditions sur le corps $\mathbf{F}$ et sur l'homomorphisme φ.

Enfin, nous restreignons le calcul de clovers en ne considérant plus que les *clovers bouclés*. C'est une restriction forte puisque J. Levine a montré que la chirurgie le long d'un clover bouclé préserve la classe de cobordisme d'homologie de la 3-variété.
Nous déduisons de ce qui précède une formule explicite décrivant comment varient les torsions abéliennes de Reidemeister-Turaev lors d'un tel mouvement chirurgical. Nous montrons alors que ces torsions, vis-à-vis de ces mouvements, sont *multiplicativement* des invariants de type fini de degré 1, comme il découle de :

THÉORÈME 9. *Soient M une 3-variété fermée orientée, $\mathbf{Z}[H_1(M)] \xrightarrow{\varphi} \mathbf{F}$ un homomorphisme d'anneaux à valeurs dans un corps commutatif $\mathbf{F}$ et G_1, G_2 des clovers de M disjoints tels que G_1 ou G_2 soit bouclé. Pour toute $\sigma \in Spin^c(M)$, nous avons alors :*

$$\tau^{\varphi_{G_1,G_2}}(M_{G_1,G_2}, \sigma_{G_1,G_2}) \cdot \tau^{\varphi}(M, \sigma) = C \cdot \tau^{\varphi_{G_1}}(M_{G_1}, \sigma_{G_1}) \cdot \tau^{\varphi_{G_2}}(M_{G_2}, \sigma_{G_2}) \in \mathbf{F},$$

où $C \in \varphi(H_1(M))$ est une constante indépendante de σ, qui est égale à 1 sous certaines conditions sur le corps $\mathbf{F}$ et sur l'homomorphisme φ.

...

Pour la lecture de cette thèse, notons que les différentes parties sont assez indépendantes les unes des autres et peuvent donc être lues séparément. Des résultats et des définitions appartenant aux chapitres numérotés 2 et 4 ont ainsi été reproduits au Chap. 1.
Les chapitres premier et dernier sont rédigés en français, les autres chapitres sont écrits en anglais.

CHAPITRE 1

Raffinements spinoriels de la théorie d'invariants de type fini de Goussarov-Habiro

1. Rappels sur la théorie de Goussarov-Habiro

Nous rappelons ici les notions essentielles de la théorie d'invariants de type fini découverte indépendamment par Mikhail Goussarov et Kazuo Habiro. Cette théorie se caractérise chez chacun des deux auteurs par la mise en place d'un calcul topologique, appelé respectivement *calcul de clovers* dans [**GGP**], et *calculs de claspers* dans [**Hr**]. Ces techniques calculatoires sont toutes deux dérivées du calcul chirurgical de Kirby, et sont en fait essentiellement équivalentes.

CONVENTIONS 1.1. Dans ce travail, nous adoptons la terminologie et les conventions de [**GGP**].

En outre, nous restreignons nos rappels à l'étude des 3-variétés, alors que la théorie de Goussarov-Habiro s'applique aussi aux entrelacs de ces dernières.

CONVENTIONS 1.2. Dans cette section, les 3-variétés sont supposées lisses, compactes et orientées.

1.1. Calcul de clovers.

1.1.1. *Y-graphes et Y-chirurgies.* Le mouvement élémentaire à partir duquel est définie la théorie de Goussarov-Habiro pour les 3-variétés, est la Y-chirurgie définie dans [**Go**].

DEFINITION 1.3. Un *Y-graphe* G dans une 3-variété M est le plongement non-orienté, dans son intérieur, de la surface dessinée sur la Figure 1.1. Cette surface est de plus décomposée en sous-surfaces appelées *feuilles* ($\cong \mathbf{S}^1 \times \mathbf{D}^1$), *côtés* ($\cong \mathbf{D}^1 \times \mathbf{D}^1$) et *sommet* ($\cong \mathbf{D}^2$).

Un Y-graphe G dans M détermine, à isotopie près, un plongement positif du corps en anses de genre 3 :

$$H_3 \xhookrightarrow{j} M,$$

d'image un voisinage régulier $N(G)$ de G dans M.

DEFINITION 1.4. La 3-variété *obtenue de M par Y-chirurgie le long de G* est :

$$M_G = M \setminus int\,(N(G)) \cup_{j|_\partial} (H_3)_L,$$

où $(H_3)_L$ est le corps en anses chirurgisé sur l'entrelacs en bande[1] à six composantes L de la Figure 1.2. On appelle *Y-équivalence*, la relation d'équivalence parmi les 3-variétés engendrée par les Y-chirurgies et les difféomorphismes positifs.

Notons que la variété M_G vient avec une inclusion $M \setminus int\,(N(G)) \hookrightarrow M$ et que, du fait du choix de $N(G)$ et de sa trivialisation j, M_G n'est définie qu'à certains difféomorphismes près.

[1]On utilise la convention du tableau.

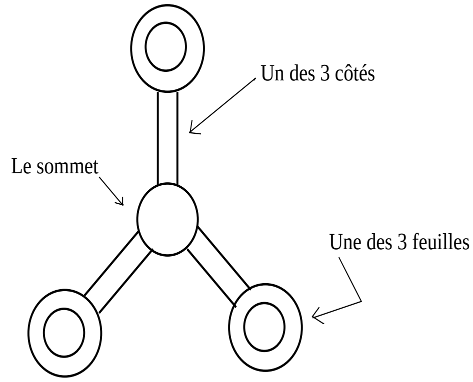

FIG. 1.1 – Surface sous-jacente à un Y-graphe.

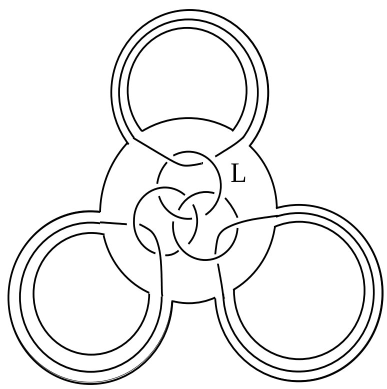

FIG. 1.2 – La signification de chirurgie d'un Y-graphe.

REMARQUE 1.5. Une Y-chirurgie équivaut à une *chirurgie Borroméenne* telle que définie par Matveev dans [**Mt**]. On en déduit[2] en particulier l'existence d'un difféomorphisme positif de $\Sigma_3 = \partial H_3$:

$$\Sigma_3 \xrightarrow[\cong]{h} \Sigma_3,$$

appelé *difféomorphisme Borroméen*, qui agit trivialement en homologie et tel qu'il existe une difféomorphisme positif :

$$M_G \cong M \setminus int\,(N(G)) \cup_{j|_\partial \circ h} H_3,$$

fixant point par point $M \setminus int\,(N(G))$.

1.1.2. *Clovers et Y_k-chirurgies.* Les Y-graphes peuvent être généralisés aux clovers définis dans [**GGP**].

DEFINITION 1.6. Un *clover* G dans une 3-variété M est le plongement non-orienté, dans son intérieur, d'une surface décomposée en sous-surfaces appelées

[2]Le lecteur pourra trouver une justification détaillée de ces faits au Chapitre 2 : Lemme 2.7 et Remarque 2.8.

feuilles ($\cong \mathbf{S}^1 \times \mathbf{D}^1$), *côtés* ($\cong \mathbf{D}^1 \times \mathbf{D}^1$) et *sommets* ($\cong \mathbf{D}^2$). Le *corps* du clover G est le complémentaire des feuilles dans la surface. On demande alors que :

- le corps soit l'épaississement d'un graphe unitrivalent, faisant correspondre respectivement les sommets et côtés de la surface, aux sommets trivalents et côtés de ce graphe ;
- la surface s'obtient par recollement du corps avec les feuilles le long de l'épaississement des sommets univalents ;
- la surface possède au moins un sommet.

Le *degré* du clover G est le nombre de ses sommets : on le note $deg(G) \geq 1$. Le *degré de boucles* du clover G est le rang du groupe fondamental de son corps (ce-dernier étant un groupe libre) : on le note $l\text{-}deg(G) \geq 0$.

EXEMPLE 1.7. Un clover de degré 1 et de degré de boucles 0 n'est autre qu'un Y-graphe.

Un clover est donc avant tout une surface plongée dans une variété de dimension 3. Il s'obtient par épaississement d'un objet 1-dimensionnel (son corps étant par définition l'épaississement d'un graphe unitrivalent, et les feuilles étant l'épaississement de cercles). En particulier, pour dessiner un clover (dans $\mathbf{S}^3$ ou dans le corps en anses H_g), on peut adopter *la convention du tableau* : nous ne dessinerons que ces objets 1-dimensionnels, qui devront être épaissis mentalement dans le plan de la feuille. De plus, les sommets du clover seront dessinés en gras afin de les distinguer d'un point d'intersection "feuille/côté".

EXEMPLE 1.8. Est représenté sur la Figure 1.3 un clover G dans H_4. Son degré est 4 et son degré de boucles est 2 (noter en effet la présence d'un côté bouclé).

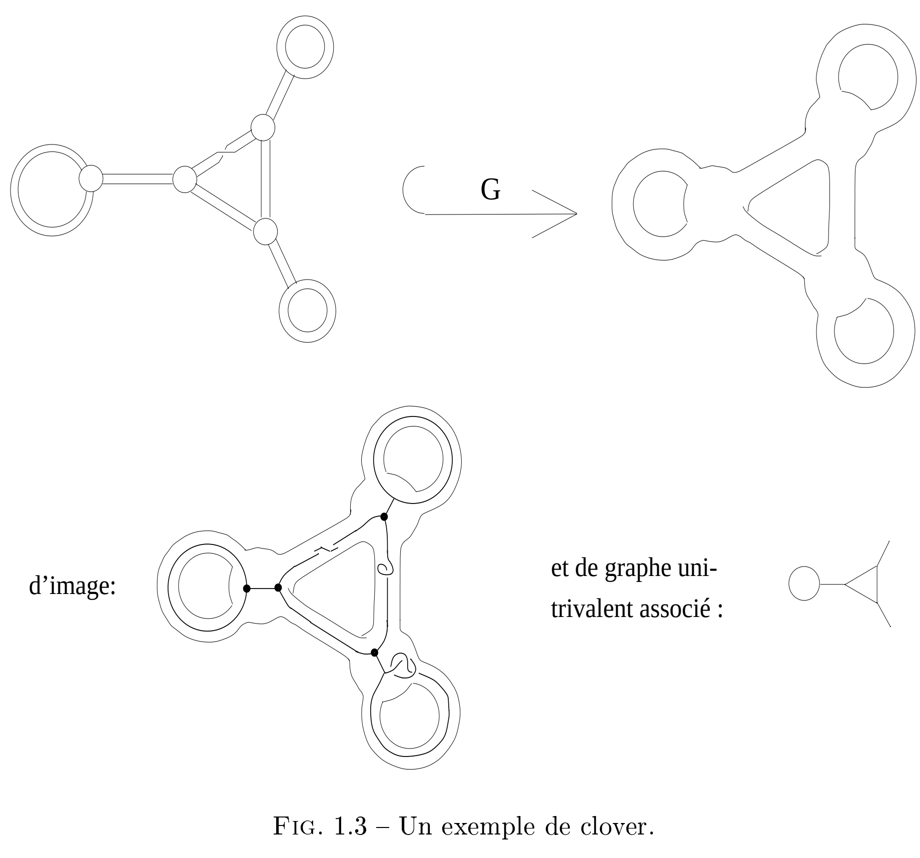

FIG. 1.3 – Un exemple de clover.

Soit G un clover de degré d dans une 3-variété M. On note alors :

$$Y(G) \subset M,$$

la réunion disjointe des d Y-graphes contenus dans un voisinage régulier $N(G)$ de G dans M, et obtenus de G en modifiant chaque côté joignant deux sommets, suivant la règle illustrée sur la Figure 1.4.

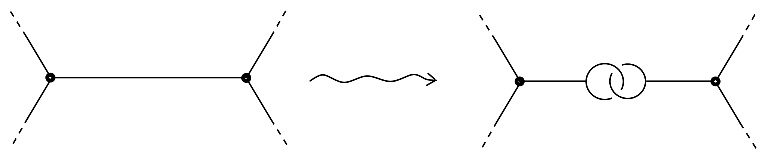

FIG. 1.4 – Fission d'un clover.

DEFINITION 1.9. La 3-variété obtenue de M par *chirurgie le long du clover* G, notée M_G, est la variété M chirurgisée sur chacun des Y-graphes que compte la famille $Y(G)$.
Pour $k \geq 1$, une *Y_k-chirurgie* est une chirurgie sur un clover de degré k. On appelle *Y_k-équivalence* la relation d'équivalence parmi les 3-variétés engendrée par les difféomorphismes positifs et les Y_k-chirurgies.

Notons que la variété M_G peut être accompagnée d'une inclusion $M \backslash int\,(N(G)) \hookrightarrow M_G$.
Observons aussi que, lorsque ∂M est non-vide, la chirurgie sur le clover G induit une identification canonique $\partial M \xrightarrow{\cong} \partial M_G$.

DEFINITION 1.10. Soient G_1 et G_2 des clovers dans une 3-variété M. On dit que les clovers G_1 et G_2 sont *équivalents*, et on note $G_1 \sim G_2$, s'il existe un corps en anses H plongé dans $int(M)$ dont l'intérieur contient à la fois G_1 et G_2, et s'il existe un difféomorphisme positif $H_{G_1} \xrightarrow[\cong]{f} H_{G_2}$ appelé *équivalence de clovers*, dont la restriction au bord soit compatible avec les identifications canoniques :

$$\begin{tikzcd}
\partial H_{G_1} \arrow[rr, "f|", "\cong"'] & & \partial H_{G_2} \\
& \partial H \arrow[ul, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

En particulier, $G_1 \sim G_2$ entraîne $M_{G_1} \cong M_{G_2}$. Précisément, *via* les identifications canoniques :

$$M_{G_i} \cong (M \setminus int(H)) \ \cup \ H_{G_i}$$

l'équivalence f induit le difféomorphisme $M_{G_1} \xrightarrow{\tilde{f}} M_{G_2}$ défini par le recollement :

$$\tilde{f} = Id_{M \setminus int(H)} \ \cup \ f.$$

Le *calcul de clovers* est un corpus de règles de calcul qui, exprimées diagrammatiquement, énoncent des équivalences de clovers. Le calcul de clovers permet donc démontrer l'existence de difféomorphismes positifs entre 3-variétés lorsque celles-ci sont présentées par chirurgie le long de clovers. Le lecteur désireux de découvrir cette technique est renvoyé à [**GGP**] ou à [**Hr**]. L'exemple suivant est donné à titre d'illustration.

EXEMPLE 1.11. Tout clover de degré 2 est équivalent à un clover de degré 1 suivant la règle de la Figure 1.5 (*cf.* [**Hr**, Prop. 2.5, Move 10] ou [**GGP**, Preuve du Th. 3.1]). En particulier, la relation de Y_{k+1}-équivalence est plus fine que la relation de Y_k-équivalence.

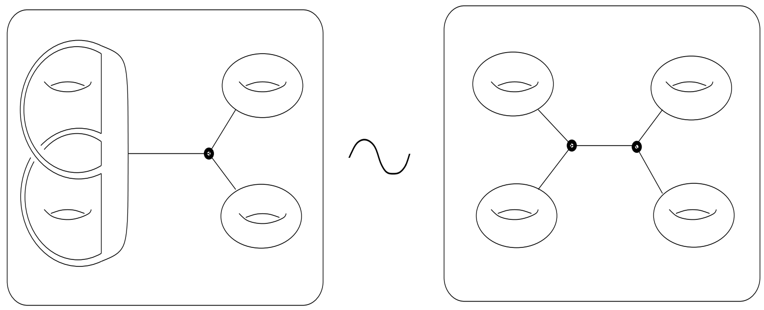

FIG. 1.5 – Une équivalence de clovers.

1.2. Invariants de type fini au sens de Goussarov et Habiro. Fixons $\mathcal{M}_0$ une classe de Y-équivalence de 3-variétés. On note alors :

$$\mathcal{F}(\mathcal{M}_0) = \mathbf{Z} \cdot \mathcal{M}_0 \tag{1.1}$$

le groupe abélien librement engendré par les 3-variétés M appartenant à $\mathcal{M}_0$.

Si M est un élément de $\mathcal{M}_0$ et si Γ y est une famille de clovers G_i deux à deux disjoints, on définit :

$$[M, \Gamma] = \sum_{\Gamma' \subset \Gamma} (-1)^{|\Gamma'|} M_{\Gamma'} \in \mathcal{F}(\mathcal{M}_0),$$

où la somme alternée est prise sur toutes les parties Γ' de Γ, de $\varnothing$ à Γ, et où $M_{\Gamma'}$ désigne la variété obtenue de M par chirurgie le long de chacun des clovers de la sous-famille Γ'. Une telle famille Γ, en plus de son cardinal $|\Gamma|$, possède un degré $deg(\Gamma)$ défini comme la somme des degrés de ses éléments. Pour des entiers $l \leq k$, ce crochet permet de construire les sous-groupes abéliens suivants de $\mathcal{F}(\mathcal{M}_0)$:

$$\begin{aligned}
\mathcal{F}_k^l(\mathcal{M}_0) &= \langle\ [M, \Gamma]\ :\ M \in \mathcal{M}_0,\ deg(\Gamma) = k,\ |\Gamma| = l\ \rangle\,, \\
\mathcal{F}_k(\mathcal{M}_0) &= \langle\ [M, \Gamma]\ :\ M \in \mathcal{M}_0,\ deg(\Gamma) = k\ \rangle\,, \\
\mathcal{F}^l(\mathcal{M}_0) &= \langle\ [M, \Gamma]\ :\ M \in \mathcal{M}_0,\ |\Gamma| = l\ \rangle\,.
\end{aligned}$$

Notons qu'alors : $\mathcal{F}_k(\mathcal{M}_0) = \bigcup_{l \leq k} \mathcal{F}_k^l(\mathcal{M}_0)$ et $\mathcal{F}^l(\mathcal{M}_0) = \bigcup_{k \geq l} \mathcal{F}_k^l(\mathcal{M}_0)$.

Pour des entiers $1 \leq l \leq l' \leq k \leq k'$, on vérifie par des calculs élémentaires de clovers :

$$\mathcal{F}_{k'}^l(\mathcal{M}_0) \subset \mathcal{F}_k^l(\mathcal{M}_0) \subset \mathcal{F}_k^{l'}(\mathcal{M}_0).$$

On en déduit en particulier que :

$$\mathcal{F}_k(\mathcal{M}_0) = \mathcal{F}_k^k(\mathcal{M}_0) = \mathcal{F}^k(\mathcal{M}_0). \tag{1.2}$$

De plus, si Γ est une famille de Y-graphes de M deux à deux disjoints et disjoints d'un Y-graphe G dans M on obtient formellement cette égalité :

$$[M, \Gamma \cup \{G\}] = [M, \Gamma] - [M_G, \Gamma].$$

On en déduit que la famille $\left(\mathcal{F}_k^k(\mathcal{M}_0)\right)_{k \geq 1}$ est une filtration descendante de $\mathcal{F}(\mathcal{M}_0)$:

$$\mathcal{F}(\mathcal{M}_0) := \mathcal{F}_0(\mathcal{M}_0) \supset \mathcal{F}_1(\mathcal{M}_0) \supset \cdots \supset \mathcal{F}_k(\mathcal{M}_0) \supset \mathcal{F}_{k+1}(\mathcal{M}_0) \supset \cdots$$

Nous pouvons maintenant énoncer la notion d'invariant de type fini selon Goussarov et Habiro :

Definition 1.12. Soit A un groupe abélien. Une application $\mathcal{M}_0 \xrightarrow{f} A$ est un *invariant de type fini de degré au plus* k, si son extension $\mathcal{F}(\mathcal{M}_0) \xrightarrow{f} A$ s'annule sur $\mathcal{F}_{k+1}(\mathcal{M}_0)$. f est un *invariant de type fini de degré* k, si son extension s'annule sur $\mathcal{F}_{k+1}(\mathcal{M}_0)$ mais prend une valeur non-nulle sur $\mathcal{F}_k(\mathcal{M}_0)$.

Remarque 1.13. Si M est Y_{k+1}-équivalente à M', alors $M - M' \in \mathcal{F}^1_{k+1}(\mathcal{M}_0)$, donc M et M' ne sont pas distinguées par les invariants de type fini de degré k.

Pour A un groupe abélien fixé, l'espace des invariants $\mathcal{M}_0 \longrightarrow A$ de degré au plus k s'identifie donc à :

$$Hom\left(\frac{\mathcal{F}(\mathcal{M}_0)}{\mathcal{F}_{k+1}(\mathcal{M}_0)}, A\right).$$

Donc, l'étude des invariants de type fini de degré au plus k est ramenée à celle de du quotient :

$$\frac{\mathcal{F}(\mathcal{M}_0)}{\mathcal{F}_{k+1}(\mathcal{M}_0)}.$$

L'examen de la théorie d'invariants de type fini de Goussarov-Habiro *se scinde* donc en deux parties de natures très différentes :

(1) reconnaître les classes de Y-équivalence,

(2) pour chaque classe de Y-équivalence $\mathcal{M}_0$, étudier la sous-théorie correspondante $(\mathcal{F}(\mathcal{M}_0)/\mathcal{F}_{k+1}(\mathcal{M}_0))_{k\geq 0}$.

Dans le cas de variétés fermées, le problème (1) de la caractérisation des invariants de degré 0 est complètement résolu. En effet, comme mentionné dans la Remarque 1.5, la Y-équivalence est engendrée par les chirurgies Borroméennes de Matveev. Pour les 3-variétés fermées, cette relation d'équivalence chirurgicale est caractérisée.

Théorème 1.14 (Matveev, [**Mt**]). *Soient M et M' des 3-variétés fermées connexes. Alors, M et M' sont Y-équivalentes si, et seulement si, il existe un isomorphisme d'homologie $H_1(M) \xrightarrow{\psi} H_1(M')$ qui fasse correspondre leurs formes d'enlacement de torsion :*

$$\begin{tikzcd} TH_1(M) \otimes TH_1(M) \arrow[r, "\lambda_M"] \arrow[d, "\psi\otimes\psi"', "\simeq"] & \frac{\mathbf{Q}}{\mathbf{Z}} \\ TH_1(M') \otimes TH_1(M'). \arrow[ur, "\lambda_{M'}"'] & \end{tikzcd}$$

Le problème de la caractérisation de la Y-équivalence est ainsi ramené à celui de la classification des formes bilinéaires symétriques non-dégénérées sur les groupes abéliens finis. Cette dernière, initiée par Minkowski, a été entreprise par Wall dans [**W1**] et achevée par Kawauchi et Kojima dans [**KK**].

2. Structures *Spin* et *Spin*^{c} sur les 3-variétés

Après une présentation des *Spin*-structures et des *Spin*^{c}-structures, nous nous attaquons dans cette section au délicat problème du recollement de ces structures, lequel problème nécessite la prise en compte de versions "rigides" de ces structures.

Conventions 2.1. Si G est un groupe abélien, un *G-espace affine* (ou *espace affine au-dessus de G*) sera un ensemble sur lequel G agit librement et transitivement. Pour ces actions, la notation multiplicative à gauche sera adoptée.
De plus, TG désignera le sous-groupe de torsion du groupe G.

CONVENTIONS 2.2. Pour les fibrés, nous conviendrons que :
- les fibrés vectoriels sont stabilisés par la gauche ;
- si G est un groupe, le G-fibré principal universel sera noté $EG \xrightarrow{\omega_G} BG$;
- si F est un morphisme de fibrés, l'application sous-jacente pour les espaces bases sera notée avec la lettre de bas de casse correspondante f.

CONVENTIONS 2.3. Concernant les variétés, nous conviendrons que :
- les variétés prises en compte sont lisses, compactes et orientées, et les difféomorphismes entre-elles sont supposés préserver les orientations ;
- la variété obtenue de M par inversion de son orientation sera notée $-M$;
- si M est une variété à bord, ∂M est orienté avec la convention du "premier vecteur normal sortant" ;
- si M est une variété, T_M (resp. τ_M) désignera son fibré tangent orienté (resp. stable) et, lorsque M sera équipée d'une métrique, nous noterons FM le fibré de ses repères orthonormés directs.

2.1. Généralités sur les structures *Spin* et *Spin*^{c}. La présentation générale qui suit des structures spinorielles est tirée de [**BM**] ; *mutatis mutandis* on obtient celle des structures spinorielles complexes.

2.1.1. *Groupes spinoriel et spinoriel complexe.* Le *groupe spinoriel Spin* est le revêtement double de SO :

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow Spin \xrightarrow{\pi} SO \longrightarrow 1,$$

et le *groupe spinoriel complexe Spin*^{c} est défini comme :

$$Spin^c = \frac{Spin \times U(1)}{\mathbf{Z}_2}, \tag{2.1}$$

où $\mathbf{Z}_2$ est le sous-groupe engendré par $[(-1,-1)]$; d'où cette autre suite exacte courte de groupes :

$$1 \longrightarrow U(1) \longrightarrow Spin^c \xrightarrow{\rho} SO \longrightarrow 1,$$

où ρ envoie $[(x,y)]$ sur $\pi(x)$. On en déduit les fibrations suivantes pour les espaces classifiants :

$$B\mathbf{Z}_2 \rightarrowtail BSpin \xrightarrow{B\pi} BSO,$$

$$BU(1) \rightarrowtail BSpin^c \xrightarrow{B\rho} BSO.$$

Nous notons γ_{SO} le fibré vectoriel orienté universel sur BSO, γ_{Spin} son pull-back par $B\pi$ et γ_{Spin^c} son pull-back par $B\rho$.

De façon analogue, pour $n \geq 1$, on définit à partir de $SO(n)$, le groupe $Spin(n)$ puis le groupe $Spin^c(n)$.

2.1.2. *Structures absolues.* Soit maintenant M une variété de dimension n (sans métrique spécifiée).

DEFINITION 2.4. Une *Spin-structure rigide* sur M est un morphisme de fibrés vectoriels orientés $\tau_M \longrightarrow \gamma_{Spin}$. Une *Spin-structure* (ou *structure spinorielle*) sur M est une classe d'homotopie de *Spin*-structure rigides.
On notera $Spin(M)$ l'ensemble des *Spin*-structures sur M, et $Spin_r(M)$ l'ensemble des *Spin*-structures rigides sur M.

DEFINITION 2.5. On définit similairement $Spin^c_r(M)$ l'ensemble des *Spin*^{c}-*structures rigides* sur M, et $Spin^c(M)$ l'ensemble des *Spin*^{c}-*structures* (ou *structures spinorielles complexes*) sur M.

Dans la suite, la lettre β désignera un homomorphisme de Bockstein associé à la suite exacte courte de coefficients :

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\cdot 2} \mathbf{Z} \longrightarrow \mathbf{Z}_2 \longrightarrow 0.$$

On rappelle maintenant les résultats concernant l'existence et le paramétrage de ces structures spinorielles.

Proposition 2.6. *La variété M admet une Spin-structure si et seulement si sa deuxième classe de Stiefel-Whitney :*

$$w_2(M) \in H^2(M; \mathbf{Z}_2)$$

est nulle, auquel cas Spin(M) est un $H^1(M;\mathbf{Z}_2)$-espace affine.
La variété M admet une Spin^c-structure si et seulement si la classe :

$$w(M) := \beta w_2(M) \in H^3(M)$$

est nulle, auquel cas Spin$^c(M)$ est un $H^2(M)$-espace affine.

Démonstration. Supposons que $\tau_M \xrightarrow{G} \gamma_{Spin^c}$ est une *Spin^c*-structure rigide sur M. En composant G avec le morphisme canonique $\gamma_{Spin^c} \longrightarrow \gamma_{SO}$, on obtient un morphisme $\tau_M \xrightarrow{F} \gamma_{SO}$. Au niveau des espaces bases, $M \xrightarrow{f} BSO$ est une application classifiante pour τ_M et $M \xrightarrow{g} BSpin^c$ est alors un relèvement de f par $B\rho$. Par construction de γ_{Spin^c}, la donnée d'une *Spin^c*-structure rigide G sur M *équivaut* à la donnée de (F,g) où :

(1) $\tau_M \xrightarrow{F} \gamma_{SO}$ est un morphisme de fibrés vectoriels orientés,

(2) g est un relèvement de f par $B\rho$.

Ainsi, se donner une *Spin^c*-structure sur M revient à se donner une classe d'homotopie d'un tel couple (F,g).
Supposons maintenant que F est *fixé*. L'espace de morphismes de fibrés :

$$Map_{SO}(\tau_M, \gamma_{SO})$$

est connexe par arcs et contractile (voir par exemple [**Hu**, Ch. 7, Prop. 3.3] et [**Hu**, Ch.7, Th. 3.4]). On peut alors déduire de ces deux faits que se donner une *Spin^c*-structure sur M revient à se donner un relèvement g de f par $B\rho$, à homotopie près de relèvements. Cette observation très utile nous invite à appliquer la théorie d'obstruction usuelle à la fibration $BSpin^c \xrightarrow{B\rho} BSO$. C'est une fibration principale de fibre $BU(1) \simeq BK(\mathbf{Z},1) \simeq K(\mathbf{Z},2)$, et de classe caractéristique $w := \beta w_2 \in H^3(BSO)$. L'énoncé de la proposition dans le cas *Spin^c* en découle.

Dans le cas *Spin*, la fibration est $BSpin \xrightarrow{B\pi} BSO$: elle est principale de fibre $B\mathbf{Z}_2 \simeq K(\mathbf{Z}_2, 1)$ et de classe caractéristique $w_2 \in H^2(BSO; \mathbf{Z}_2)$. □

Le lemme suivant nous assure que nos définitions numérotées 2.4 et 2.5 s'accordent avec les définitions plus communes des *Spin*-structures et des *Spin^c*-structures.

Lemme 2.7. *Supposons que M est munie d'une métrique riemannienne. La donnée d'une Spin-structure sur M est alors équivalente à une double donnée (η, H), à isomorphisme près, où η est un Spin(n)-fibré principal de base M, et où $\eta/\mathbf{Z}_2 \xrightarrow{H} FM$ est un isomorphisme de $SO(n)$-fibrés principaux.*
De même, se donner une Spin^c-structure sur M équivaut à se donner un couple (η, H), à isomorphisme près, où η est un Spin$^c(n)$-fibré principal de base M et où $\eta/U(1) \xrightarrow{H} FM$ est un isomorphisme de $SO(n)$-fibrés principaux.

DÉMONSTRATION. Intéressons-nous par exemple au cas $Spin^c$, le traitement du cas $Spin$ étant formellement le même. Soit (η, H) comme dans l'énoncé : montrons que ce couple détermine une $Spin^c$-structure rigide G sur M, en montrant qu'il détermine une donnée équivalente (F, g), telle que décrite dans la preuve de la Prop. 2.6. Choisissons un morphisme de $Spin^c(n)$-fibrés $\eta \longrightarrow \omega_{Spin^c(n)}$ qui, composé avec le morphisme canonique $\omega_{Spin^c(n)} \longrightarrow \omega_{Spin^c}$, donne un certain $\eta \xrightarrow{\tilde{G}} \omega_{Spin^c}$. Ce $\tilde{G}$ induit un morphisme $\eta/U(1) \longrightarrow \omega_{SO}$; en composant avec H^{-1}, nous obtenons $FM \xrightarrow{\tilde{F}} \omega_{SO}$. $\tilde{F}$ induit alors un morphisme $\tau_M \xrightarrow{F} \gamma_{SO}$. On pose $g := \tilde{g}$, induite par $\tilde{G}$ au niveau des espaces bases. Alors, l'assignation $(\eta, H) \longmapsto (F, g) \equiv G$ induit la correspondance annoncée entre les deux définitions de $Spin^c$-structures. □

2.1.3. *Structures relatives.* Soit M une n-variété *à bord*. Ayant convenu de stabiliser les fibrés vectoriels par la gauche et d'orienter les bords avec la règle du premier vecteur normal sortant, $\tau_{\partial M}$ peut être identifié avec $\tau_M|_{\partial M}$, d'où des applications de restrictions :

$$Spin_r(M) \xrightarrow{rest} Spin_r(\partial M) \quad \text{et :} \quad Spin_r^c(M) \xrightarrow{rest} Spin_r^c(\partial M).$$

DEFINITION 2.8. Pour $s \in Spin_r(\partial M)$ fixée, une *$Spin$-structure sur M relative à s* est une classe d'homotopie rel ∂M de $Spin$-structures rigides sur M étendant s ; on note $Spin(M, s)$ l'ensemble de ces structures relatives.
De même, pour $s \in Spin_r^c(\partial M)$ fixée, on définit l'ensemble $Spin^c(M, s)$ des *$Spin^c$-structures sur M relatives à s*.

La théorie d'obstruction peut être appliquée pour obtenir cet analogue relatif de la Proposition 2.6 :

PROPOSITION 2.9. *Soit $s \in Spin_r(\partial M)$ fixée. Alors, s peut être étendue à M si et seulement si une certaine classe :*

$$w_2(M, s) \in H^2(M, \partial M; \mathbf{Z}_2)$$

s'annule, auquel cas $Spin(M, s)$ est un $H^1(M, \partial M; \mathbf{Z}_2)$-espace affine. En outre, la restriction de $w_2(M, s)$ à M est égale à $w_2(M) \in H^2(M; \mathbf{Z}_2)$.
De même, pour $s \in Spin_r^c(\partial M)$ fixée, s peut être étendue à M si et seulement si une certaine classe :

$$w(M, s) \in H^3(M, \partial M)$$

s'annule, auquel cas $Spin^c(M, s)$ est un $H^2(M, \partial M)$-espace affine. En outre, la restriction de $w(M, s)$ à M est $w(M) = \beta w_2(M) \in H^3(M)$.

2.1.4. *Inversion des orientations.* Il existe une flèche :

$$Spin_r^c(M) \xrightarrow{-} Spin_r^c(-M),$$

induisant une application $H^2(M)$-équivariante :

$$Spin^c(M) \xrightarrow{-} Spin^c(-M).$$

Elle consiste à composer les structures rigides avec la réflexion orthogonale de τ_{-M} vers τ_M dans la direction de la première stabilisation.

Supposons maintenant $\partial M \neq \varnothing$ et soit $s \in Spin_r^c(\partial M)$. Faisons le choix d'une section non-singulière v de τ_M étendant le champ de vecteurs normal sortant sur ∂M (un tel v existe et est unique à homotopie relative près). En composant avec la réflection orthogonale de τ_{-M} vers τ_M dans la direction de v, on obtient une flèche (indépendante de v) :

$$Spin^c(M, s) \xrightarrow{-} Spin^c(-M, -s),$$

qui est $H^2(M,\partial M)$-equivariante. Le cas des *Spin*-structures est analogue.

2.1.5. *De Spin à Spin^c.* Considérons le diagramme commutatif de groupes :

$$\begin{tikzcd} Spin \arrow[r, "\beta"] \arrow[dr, "\pi"'] & Spin^c \arrow[d, "\rho"] \\ & SO \end{tikzcd}$$

où la flèche β est définie par $\beta(x) = [(x,1)]$. Alors, $B\pi = B\rho \circ B\beta$ au niveau des espaces classifiants. D'où un morphisme de fibrés : $\gamma_{Spin} \longrightarrow \gamma_{Spin^c}$, défini de la façon évidente.
Si M est une variété, il existe alors une application $Spin_r(M) \longrightarrow Spin^c_r(M)$, laquelle induit une application :

$$Spin(M) \xrightarrow{\beta} Spin^c(M),$$

qui est affine au-dessus de l'homomorphisme de Bockstein $H^1(M;\mathbf{Z}_2) \xrightarrow{\beta} H^2(M)$.
Si maintenant M est une variété de bord non-vide et si $s \in Spin_r(\partial M)$, on a une application :

$$Spin(M,s) \xrightarrow{\beta} Spin^c(M,\beta(s)),$$

qui est affine au-dessus de l'homomorphisme de Bockstein $H^1(M,\partial M;\mathbf{Z}_2) \xrightarrow{\beta} H^2(M,\partial M)$.

2.1.6. *Classes de Chern.* Soit M une n-variété et soit $\alpha \in Spin^c(M)$. Alors du fait de l'homomorphisme de groupes $Spin^c(n) \longrightarrow U(1)$ définie par : $[(x,y)] \longmapsto y^2$, la $Spin^c$-structure α, prise comme dans le Lemme 2.7, détermine (à isomorphisme près) un $U(1)$-fibré principal : soit $c(\alpha)$ sa première classe de Chern.

Definition 2.10. La classe $c(\alpha) \in H^2(M)$ est appelée *classe de Chern de la Spin^c-structure α*. Si $c(\alpha) \in TH^2(M)$, la $Spin^c$-structure α est dite *de torsion*.

Remarque 2.11. L'application classe de Chern $Spin^c(M) \xrightarrow{c} H^2(M)$, est affine au-dessus de l'application carré[3] $H^2(M) \longrightarrow H^2(M)$ définie par : $x \longmapsto x^2$. Donc, $\alpha \in Spin^c(M)$ provient de $Spin(M)$ par l'application β définie au §2.1.5 si et seulement si $c(\alpha) = 0$.

2.2. Structures en dimension 3. A partir de maintenant, nous traitons le cas spécifique de la dimension 3. Si M est une variété (compacte lisse orientée) de dimension 3, alors $w_2(M)$ est nulle. Donc, M admet des *Spin*-structures et des $Spin^c$-structures.

2.2.1. *Groupes spinoriels en dimension* 3. Dans ce paragraphe, nous rappelons les isomorphismes $Spin(3) \simeq SU(2)$ et $Spin^c(3) \simeq U(2)$ en faisant des opérations quaternioniques.

Soit donc $\mathbf{H}$ le corps des quaternions :

$$\mathbf{H} = \{q = a + b\cdot i + c\cdot j + d\cdot k : a,b,c,d \in \mathbf{R}\}$$

et notons $\mathbf{R}^3$ le sous-espace des quaternions purs $\mathbf{R}^3 = \mathbf{R}\cdot i + \mathbf{R}\cdot j + \mathbf{R}\cdot k$. Le groupe des quaternions unitaires : $\mathbf{S}^3 = \{q = a + b\cdot i + c\cdot j + d\cdot k : a^2+b^2+c^2+d^2 = 1\}$ s'identifie à $SU(2)$ par :

$$q = a + b.i + c.j + d.k \longmapsto \begin{pmatrix} a+b.i & -d+c.i \\ d+c.i & a-b.i \end{pmatrix}.$$

[3]On rappelle que la notation multiplicative a été adoptée pour les actions affines.

On définit alors l'homorphisme $SU(2) \xrightarrow{\pi} SO(3)$, qui envoie un quaternion unitaire q sur la transformation :

$$\begin{array}{ccc} \mathbf{R}^3 & \xrightarrow{\pi(q)} & \mathbf{R}^3 \\ h & \longmapsto & qhq^{-1}. \end{array}$$

π est un revêtement à deux feuilles de $SO(3)$, on peut donc poser :

$$Spin(3) = SU(2).$$

Grâce à l'isomorphisme :

$$\frac{SU(2) \times U(1)}{\mathbf{Z}_2} \xrightarrow{\simeq} U(2)$$

envoyant $[(A, z)]$ sur zA, on peut donc aussi poser :

$$Spin^c(3) = U(2).$$

REMARQUE 2.12. L'homomorphisme canonique $Spin(3) \xrightarrow{\beta} Spin^c(3)$ (voir §2.1.5) correspond à l'inclusion standard $SU(2) \hookrightarrow U(2)$. La suite exacte courte mettant en jeu les groupes $U(1)$, $Spin^c(3)$ et $SO(3)$ (§2.1.1) s'écrit :

$$1 \longrightarrow U(1) \longrightarrow U(2) \xrightarrow{\rho} SO(3) \longrightarrow 1$$

où l'inclusion de $U(1)$ dans $U(2)$ consiste à envoyer z sur $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$, et où $\rho(A) = \pi\left(\frac{1}{\sqrt{det(A)}} \cdot A\right)$. Enfin, l'homomorphisme canonique $Spin^c(3) \longrightarrow U(1)$ (§2.1.6) est l'application déterminant $U(2) \xrightarrow{det} U(1)$.

LEMME 2.13. *Le diagramme suivant est commutatif :*

$$\begin{tikzcd} SO(2) \arrow[r, "\simeq"] \arrow[d, tail] & U(1) \arrow[r, tail] & U(2) \arrow[d, "\rho"] \\ SO(3) \arrow[rr, equal] & & SO(3). \end{tikzcd}$$

Ici, $SO(2)$ s'injecte dans $SO(3)$ par $A \longmapsto (1) \oplus A$, et $U(1)$ s'injecte dans $U(2)$ par : $A \longmapsto A \oplus (1)$.

DÉMONSTRATION. Soit $z = e^{i\theta} \in U(1)$ identifié avec :

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in SO(2).$$

Si on envoie z dans $U(2)$ puis dans $SO(3)$, on obtient $\pi(A)$ où :

$$A = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \in SU(2).$$

Mais, A s'identifie au quaternion pur : $q = \cos(\theta/2) + \sin(\theta/2) \cdot i$. On calcule alors que :

$$q \cdot (bi + cj + dk) \cdot q^{-1} = bi + c\cos(\theta)j + d\cos(\theta)k + c\sin(\theta)k - d\sin(\theta)j.$$

Donc,

$$\pi(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

□

2.2.2. *Structures relatives en dimension* 3. Nous avons introduit plus haut les *Spin*-structures rigides et les $Spin^c$-structures rigides. Ces structures rigides apparaissent naturellement avec la théorie d'homotopie, mais n'ont cependant pas d'existence légitime d'un point de vue géométrique. Ce manque de légitimité va donc se ressentir aussi pour les structures relatives et les notions qui s'y réfèrent. Dans ce paragraphe, nous corrigeons partiellement ce défaut en dimension 3, et commençons par traiter le cas $Spin^c$.

Théorème 2.14. *Soient M une 3-variété de bord non-vide et $s_0, s_1 \in Spin_r(\partial M)$ représentant la même structure spinorielle : $[s_0] = [s_1] \in Spin(\partial M)$. Alors, les $Spin^c$-structures rigides $\beta(s_i)$ peuvent être étendues à la variété M et il existe une bijection canonique :*

$$Spin^c(M, \beta s_0) \xrightarrow{\rho_{s_0,s_1}} Spin^c(M, \beta s_1),$$

qui est $H^2(M, \partial M)$-équivariante.

Definition 2.15. Le Théorème 2.14 nous permet d'associer à toute 3-variété M de bord non-vide et à toute $\sigma \in Spin(\partial M)$, l'espace :

$$Spin^c(M, \sigma)$$

des *$Spin^c$-structures sur M relatives à la structure spinorielle σ*. C'est un $H^2(M, \partial M)$-espace affine.

Exemple 2.16. Intéressons-nous au cas particulier où ∂M est une réunion disjointe de tores. Le tore $\mathbf{T}^2$ a une structure spinorielle préférée : celle induite par sa structure de groupe de Lie, nous la noterons σ_0. Donc, ∂M possède $\dot\cup\sigma_0$ comme structure spinorielle préférée. On vérifie que l'espace $Spin^c(M, \dot\cup\sigma_0)$ donné par la Définition 2.15 est en bijection canonique avec l'espace de $Spin^c$-structures relatives défini par Turaev dans [**T6**, §1.2].

Preuve du Théorème 2.14. L'obstruction $w_2(M, s_i) \in H^2(M, \partial M; \mathbf{Z}_2)$ pour étendre la *Spin*-structure rigide s_i à toute la variété M, vérifie :

$$\beta(w_2(M, s_i)) = w(M, \beta s_i) \in H^3(M, \partial M),$$

où $w(M, \beta s_i)$ est l'obstruction définie par la Proposition 2.9. On en déduit que $w(M, \beta s_i)$ est d'ordre au plus 2 et que donc elle s'annule.

On démontre maintenant la deuxième assertion. Soit $(s_t)_{t \in I}$ une homotopie entre s_0 et s_1 : s_t est un morphisme de fibrés $\tau_{\partial M} \longrightarrow \gamma_{Spin}$. Elle définit une *Spin*-structure rigide $\tau_{\partial M \times I} \xrightarrow{s} \gamma_{Spin}$ sur $\partial M \times I$ en posant :

$$s|_{(\tau_{\partial M \times I})|_{\partial M \times t}} := s_t,$$

où $\tau_{\partial M \times I}|_{\partial M \times t}$ est identifié avec $\tau_{\partial M}$. On définit alors l'application :

$$Spin^c(M, \beta s_0) \xrightarrow{\rho_s} Spin^c(M, \beta s_1)$$

par la formule de recollement : $\rho_s([u]) = [u \cup \beta s]$, où u est une $Spin^c$-structure rigide sur M étendant $\beta(s_0)$ (nous identifions ici M avec son "débordement" $M \cup (\partial M \times I)$). Remarquons que ρ_s est $H^2(M, \partial M)$-équivariante.
Pour prouver le théorème, il suffit de vérifier que si $(s'_t)_{t \in I}$ est une autre homotopie entre s_0 et s_1, alors $\rho_s = \rho_{s'}$; on posera alors $\rho_{s_0,s_1} := \rho_s$. L'image de l'application

$$Spin(\partial M \times I, (-s_0) \times 0 \cup s_1 \times 1) \xrightarrow{\beta} Spin^c(\partial M \times I, (-\beta s_0) \times 0 \cup (\beta s_1) \times 1)$$

est un singleton, puisqu'elle est affine au-dessus de :

$$H^1(\partial M \times I, \partial M \times \partial I; \mathbf{Z}_2) \xrightarrow{\beta} H^2(\partial M \times I, \partial M \times \partial I)$$

qui est triviale vu que son codomaine est isomorphe au groupe $H_1(\partial M)$ libre de torsion. On en déduit que que les *Spin*^{c}-structures rigides βs et $\beta s'$ sur $\partial M \times I$ sont homotopes rel $\partial M \times \partial I$, et que donc $\rho_s = \rho_{s'}$. □

Dans le cas *Spin*, les structures relatives que nous retiendrons *in fine*, seront les suivantes.

DEFINITION 2.17. Soient M une 3-variété de bord non-vide et $\sigma \in Spin(\partial M)$ telle que $w_2(M, \sigma) = 0 \in H^2(M, \partial M; \mathbf{Z}_2)$. Alors,

$$Spin(M, \sigma) := \{\alpha \in Spin(M) \ : \ \alpha|_{\partial M} = \sigma\}.$$

est appelé l'espace des *structures spinorielles sur M relatives à σ.*

LEMME 2.18. *Soient M une 3-variété de bord non-vide, $\sigma \in Spin(M)$ telle que $w_2(M, \sigma) = 0$ et $H^0(\partial M; \mathbf{Z}_2) \xrightarrow{\delta^*} H^1(M, \partial M; \mathbf{Z}_2)$ le connectant associé à la paire $(M, \partial M)$. Alors :*

(1) *$Spin(M, \sigma)$ est un $Coker(\delta^*)$-espace affine s'identifiant canoniquement à $Spin(M, s)/Im(\delta^*)$ pour toute $s \in Spin_r(\partial M)$ représentant σ ;*
(2) *en particulier, lorsque ∂M est connexe, $Spin(M, \sigma)$ est un $H^1(M, \partial M; \mathbf{Z}_2)$--espace affine s'identifiant canoniquement à $Spin(M, s)$ pour toute $s \in Spin_r(\partial M)$ représentant σ ;*
(3) *enfin, il existe une application canonique :*
$$Spin(M, \sigma) \xrightarrow{\beta} Spin^c(M, \sigma),$$
qui est affine au-dessus de $Coker(\delta^*) \xrightarrow{\beta} H^2(M, \partial M)$.

DÉMONSTRATION. De la suite exacte :

$$H^0(\partial M; \mathbf{Z}_2) \xrightarrow{\delta^*} H^1(M, \partial M; \mathbf{Z}_2) \xrightarrow{i^*} H^1(M; \mathbf{Z}_2),$$

on déduit que $Coker(\delta^*)$ se plonge dans $H^1(M; \mathbf{Z}_2)$. L'action affine de $Coker(\delta^*)$ sur $Spin(M, \sigma)$ est la restriction de celle de $H^1(M; \mathbf{Z}_2)$ sur $Spin(M)$. Ensuite, l'application canonique $Spin(M, s) \longrightarrow Spin(M)$ a pour image $Spin(M, \sigma)$ et est affine au-dessus de $H^1(M, \partial M; \mathbf{Z}_2) \xrightarrow{i^*} H^1(M; \mathbf{Z}_2)$. $Spin(M, \sigma)$ s'identifie donc à $Spin(M, s)$ modulo $Im(\delta^*) = Ker(i^*)$. Ceci prouve (1) et (2). Enfin, par ce diagramme commutatif :

$$\begin{array}{ccc} H^0(\partial M; \mathbf{Z}_2) & \xrightarrow{\beta} & H^1(\partial M) \\ \downarrow{\scriptstyle \delta^*} & & \downarrow{\scriptstyle \delta^*} \\ H^1(M, \partial M; \mathbf{Z}_2) & \xrightarrow{\beta} & H^2(M, \partial M) \end{array}$$

et par le fait que $H^1(\partial M)$ est libre de torsion, on déduit que $\beta(Im(\delta^*)) = 0$. Ainsi, l'application canonique $Spin(M, s) \xrightarrow{\beta} Spin^c(M, \beta s) \simeq Spin^c(M, \sigma)$, qui est affine au dessus de $H^1(M, \partial M; \mathbf{Z}_2) \xrightarrow{\beta} H^2(M, \partial M)$, induit une application équivariante entre $Spin(M, \sigma)$ et $Spin^c(M, \sigma)$, prouvant ainsi (3). □

2.2.3. *Spin^c-structures comme champs de vecteurs : le cas fermé.* Soit M une variété fermée de dimension 3. Turaev a décrit dans [**T5**] comment, en dimension 3, les *Spin*^{c}-structures pouvaient être identifiées avec certaines classes d'homotopie de champs de vecteurs tangents à la variété M. Nous rappelons ici cette correspondance.

Definition 2.19 (Turaev, [**T4**]). Une *structure d'Euler géométrique* sur M est un champ de vecteurs non-singulier sur M, à homotopie épointée près. Précisément, deux champs de vecteurs v et v' sont considérés comme équivalents, lorsqu'il existe un point x de M tel que les restrictions de v et v' sur $M \setminus x$ sont homotopes parmi les champs de vecteurs non-singuliers sur $M \setminus x$. On note :

$$Vect(M)$$

l'ensemble des structures d'Euler géométriques sur M.

Si une décomposition cellulaire de M est donnée, la relation d'homotopie épointée coincide avec la relation d'homotopie sur le 2-squelette de M. De la théorie d'obstruction, on déduit que $Vect(M)$ est non-vide (vu que $\chi(M) = 0$) et que $Vect(M)$ est un $H^2(M)$-espace affine.

Lemme 2.20 (Turaev, [**T5**]). *Il existe une bijection canonique et $H^2(M)$-équivariante :*

$$Vect(M) \underset{\simeq}{\xrightarrow{h_M}} Spin^c(M).$$

Démonstration. Soit v un champ de vecteurs non-singulier sur M. Munissons M d'une métrique riemannienne. Remarquons tout d'abord que v détermine une réduction de FM à $SO(2)$ pour cette injection $SO(2) \hookrightarrow SO(3)$ apparaissant dans le Lemme 2.13 : cette réduction est $Fv^{\perp}$, le fibré des repères orthonormés directs du fibré vectoriel $v^{\perp}$, lorsque celui-ci est orienté avec la règle de la main droite (v=pouce droit). Alors, par l'homomorphisme $SO(2) \longrightarrow U(2)$ du Lemme 2.13, $Fv^{\perp}$ definit un $U(2)$-fibré principal η. D'après ce même lemme, ce $U(2)$-fibré peut-être accompagné d'un isomorphisme de $SO(3)$-fibrés principaux $\eta/U(1) \xrightarrow{H} FM$, et définit donc (d'après le Lemme 2.7) une $Spin^c$-structure sur M. Celle-ci ne dépend que de la classe d'homotopie épointée de v : ainsi est définie $h_M([v])$.
De plus, on vérifie que l'assignation $[v] \longmapsto h_M([v])$ est $H^2(M)$-équivariante ; elle est donc en particulier bijective. □

On définit une involution de $Vect(M)$ en associant à toute structure d'Euler géométrique $\xi = [v]$ la structure $\xi^{-1} := [-v]$. Nous rappelons que nous avons convenu de noter multiplicativement les actions affines.

Lemme 2.21. *Pour toute $\xi \in Vect(M)$, nous avons :*

$$c\left(h_M(\xi)\right) = \frac{\xi}{\xi^{-1}} \in H^2(M).$$

Démonstration. Les applications $Vect(M) \longrightarrow H^2(M)$ qui sont définies par $\xi \longmapsto \xi/\xi^{-1}$ et par $\xi \longmapsto c(h_M(\xi))$, sont toutes deux affines au-dessus de l'application carré (d'après respectivement [**T4**, Th. 5.3.1] et la Remarque 2.11). Il suffit donc de montrer l'implication suivante :

$$(2.2) \qquad \left(c\left(h_M(\xi)\right) = 1 \right) \implies \left(\xi/\xi^{-1} = 1 \right).$$

Munissons M d'une métrique. D'après la Remarque 2.12, la classe d'isomorphisme de $U(1)$-fibrés principaux définis par la $Spin^c$-structure $h_M(\xi)$, admet $Fv^{\perp}$ comme représentant, si bien que $c\left(h_M(\xi)\right) = c_1\left(Fv^{\perp}\right)$. Donc, la classe $c\left(h_M(\xi)\right)$ est l'obstruction pour trouver une section non-singulière de T_M transverse à v. On en déduit que l'assertion (2.2) est vraie. □

Dans cette dernière preuve, il apparaît que l'image de $Spin(M)$ dans $Spin^c(M)$ correspond aux structures d'Euler $\xi = [v]$ pour lesquelles il existe un champ de vecteurs non-singulier transverse à v. Précisons ce fait.

DEFINITION 2.22. Nous noterons :

$$Parall(M)$$

l'ensemble des parallélisations de M, considérées à homotopie épointée près.

Par *parallélisation de* M, nous entendons une trivialisation $e = (e_1, e_2, e_3)$ du fibré vectoriel orienté T_M. En dimension ≥ 3, l'espace des trivialisations de T_M sur le 1-squelette de M, qui ont la propriété de s'étendre au 2-squelette, et considérées à homotopie près, est par la théorie d'obstruction vide ou paramétré par $H^1(M; \mathbf{Z}_2)$, et peut être mis en correspondance canonique avec $Spin(M)$ (voir [**Ki**, Chap. 4]). Puisque $\pi_2(SO(3)) = 0$, nous obtenons :

LEMME 2.23. *Il existe une bijection canonique et $H^1(M; \mathbf{Z}_2)$-équivariante :*

$$Parall(M) \xrightarrow{h_M} Spin(M).$$

DÉMONSTRATION. La preuve est similaire à celle du Lemme 2.20. Munissons M d'une métrique riemannienne. Soit e une trivialisation de T_M, ce e détermine une réduction de FM au groupe trivial $1 \hookrightarrow SU(2)$, et permet donc de construire un $SU(2)$-fibré principal η (trivial) ainsi qu'un isomorphisme de $SO(3)$-fibrés principaux $\eta/\mathbf{Z}_2 \xrightarrow{H} FM$. La *Spin*-structure correspondante (donnée par le Lemme 2.7) sur M ne dépend que de la classe d'homotopie épointée de e, et est retenue pour être $h_M([e])$. L'application ainsi construite $Parall(M) \longrightarrow Spin(M)$ est $H^1(M; \mathbf{Z}_2)$-équivariante, et est donc en particulier bijective. $\square$

En posant $\beta([e]) = [e_1]$, où $e = (e_1, e_2, e_3)$ est une parallélisation de M, on définit une application :

$$Parall(M) \xrightarrow{\beta} Vect(M).$$

Il découle alors directement des définitions que :

LEMME 2.24. *Le diagramme suivant est commutatif :*

$$\begin{tikzcd} Parall(M) \arrow[r, "h_M"] \arrow[d, "\beta"'] & Spin(M) \arrow[d, "\beta"] \\ Vect(M) \arrow[r, "h_M"'] & Spin^c(M). \end{tikzcd}$$

où $Spin(M) \xrightarrow{\beta} Spin^c(M)$ a été définie en 2.1.5.

2.2.4. *Spin*^{c}-structures comme champs de vecteurs : le cas à bord. Nous souhaitons maintenant définir des structures d'Euler géométriques pour les variétés à bord et donner ainsi une version relative du Lemme 2.20 mettant en jeu les *Spin*^{c}-structures relatives introduites dans la Définition 2.15.

D'une part, le précédent paragraphe, plutôt qu'à la variété fermée M, était en fait relatif à son fibré tangent T_M, et vaut pour n'importe quel fibré vectoriel réel orienté de dimension 3. En particulier, si S est une surface fermée, il s'applique au fibré tangent de S stabilisé une fois. On définit ainsi :

$$Vect(S_{st}) \quad \text{et} \quad Parall(S_{st}),$$

comme étant respectivement l'ensemble des sections non-singulières de $\epsilon^1 \oplus T_S$ et celui des trivialisations de ce fibré vectoriel orienté, toutes étant en cette dimension considérées à homotopie près sur S. La théorie d'obstruction nous dit qu'ils sont respectivement des espaces affines au-dessus de $H^2(S)$ et de $H^1(S; \mathbf{Z}_2)$. Les analogues des lemmes numérotes 2.20 et 2.23 assurent l'existence d'isomorphismes affines :

$$Vect(S_{st}) \xrightarrow{h_S} Spin^c(S) \quad \text{et :} \qquad Parall(S_{st}) \xrightarrow{h_S} Spin(S).$$

Enfin, l'analogue du Lemme 2.24 met en relation ces deux isomorphismes.

EXEMPLE 2.25. La section canonique $v = (1,0)$ de $\epsilon^1 \oplus T_S$ définit une $Spin^c$-structure $h_S([v])$ sur S, dont la classe de Chern est égale à la classe d'Euler $e(S)$ de la surface S (puisqu'ici $v^\perp = T_S \leq \epsilon^1 \oplus T_S$ pour une métrique produit).

D'autre part, on peut parler de structures rigides pour n'importe quel type de structures définies comme des classes d'homotopie d'applications. Il y a ainsi des versions rigides de $Vect(N)$ et de $Parall(N)$, lorsque $N = M$ pour une 3-variété M ou lorsque $N = S_{st}$ pour une surface S. Ces versions rigides sont notées avec la décoration "r" en indice inférieur.
Ainsi, si M est une 3-variété de bord non-vide et si $v \in Vect_r\left((\partial M)_{st}\right)$ est une section non-singulière de $\epsilon^1 \oplus T_{\partial M} = T_M|_{\partial M}$, on sait définir :

$$Vect(M, v)$$

l'espace des *structures d'Euler géométriques sur M relatives à v.* Cet espace est vide ou bien est un $H^2(M, \partial M)$-espace affine. Le théorème suivant se prouve alors comme le Théorème 2.14.

THÉORÈME 2.26. *Soient M une 3-variété à bord et $e = (e_1, e_2, e_3), e' = (e'_1, e'_2, e'_3)$ $\in Parall_r(\partial M_{st})$ des trivialisations de $\epsilon^1 \oplus T_{\partial M}$, qu'on suppose homotopes. Alors, les sections non-singulières e_1 et e'_1 de $T_M|_{\partial M}$ peuvent être étendues à toute la variété M, et il existe une bijection canonique :*

$$Vect(M, e_1) \xrightarrow{\rho_{e,e'}} Vect(M, e'_1),$$

qui est $H^2(M, \partial M)$-équivariante.

DEFINITION 2.27. Le Théorème 2.26 nous permet donc d'associer à toute 3-variété M de bord non-vide et à toute $\rho \in Parall\big((\partial M)_{st}\big)$, l'espace :

$$Vect(M, \rho)$$

des *structures d'Euler géométriques sur M relatives à ρ.* C'est un $H^2(M, \partial M)$-espace affine.

D'où cette version relative du Lemme 2.20 qui se prouve de façon similaire :

LEMME 2.28. *Soit M une 3-variété à bord et soit $\rho \in Parall((\partial M)_{st})$. Il existe alors une bijection canonique et $H^2(M, \partial M)$-équivariante :*

$$Vect(M, \rho) \longrightarrow Spin^c\left(M, h_{\partial M}(\rho)\right).$$

2.2.5. *Classes de Chern relatives.* Nous donnons maintenant un analogue relatif des classes de Chern de $Spin^c$-structures (rappelées au §2.1.6).

LEMME 2.29. *Soient M une 3-variété à bord et $\sigma \in Spin(\partial M)$. Il existe une application canonique :*

$$Spin^c(M, \sigma) \xrightarrow{c} H^2(M, \partial M),$$

qui est affine au-dessus de l'application carré[4] *définie par $x \longmapsto x^2$.*

DEFINITION 2.30. Pour $\alpha \in Spin^c(M, \sigma)$, $c(\alpha)$ est appelée la *classe de Chern de la $Spin^c$-structure relative α.*

[4] On rappelle que la notation multiplicative a été adoptée pour les actions affines.

DÉMONSTRATION DU LEMME 2.29. Soit $\rho \in Parall\left((\partial M)_{st}\right)$ correspondant à σ par $h_{\partial M}$. Nous allons en fait définir une application $Vect(M,\rho) \xrightarrow{c} H^2(M,\partial M)$ (puis appliquer le Lemme 2.28).

Soit $e = (e_1, e_2, e_3)$ une trivialisation de $\epsilon^1 \oplus T_{\partial M}$ représentant ρ et soit $\xi \in Vect(M, e_1)$ représenté par v : v est un champ de vecteurs non-singulier sur M dont la restriction à ∂M coïncide avec e_1. Le champ de vecteurs e_2 est une section non-singulière de T_M transverse à v définie sur ∂M. On définit alors $c(\xi)$ comme étant la première obstruction pour étendre e_2 à une section de T_M transverse à v. On obtient ainsi une application $Vect(M, e_1) \longrightarrow H^2(M, \partial M)$. Si e' est un autre représentant de ρ, l'application $Vect(M, e'_1) \longrightarrow H^2(M, \partial M)$ obtenue similairement coïncide avec la précédente *via* l'isomorphisme $\rho_{e,e'}$ du Théorème 2.26. On conclut à la bonne définition de l'application c.

Soit maintenant $x \in H^2(M, \partial M)$ et supposons que son dual de Poincaré $P^{-1}x \in H_1(M)$ est représenté par le noeud lisse orienté $L \subset int(M)$. Alors, comme dans le cas fermé, on montre que la structure d'Euler géométrique $x \cdot \xi$ est représenté par un champ de vecteurs w obtenu de v par *turbulentisation de Reeb* le long de L (voir [**T4**, §5.2] ou [**T8**, §1.1]). Par un calcul direct d'obstruction dans la 3-variété orientée M (profitant de la dualité de Poincaré), on montre à partir de cette description concrète de w que $c([w]) = x^2 \cdot c([v])$. □

REMARQUE 2.31. Pour $\xi \in Spin^c(M, \sigma)$, sa classe de Chern $c(\xi)$ s'annule si et seulement si ξ provient de $Spin(M, \sigma)$.

LEMME 2.32. *Soit* $M \xrightarrow[\simeq]{h} M'$ *un difféomorphisme entre* 3*-variétés à bord, et soit* $\sigma' \in Spin(\partial M')$. *Alors,*

$$\forall \alpha' \in Spin^c(M', \sigma'), \quad c\left(h^*\left(\alpha'\right)\right) = h^*\left(c\left(\alpha'\right)\right) \in H^2(M, \partial M).$$

DÉMONSTRATION. Soit $e' = (e'_1, e'_2, e'_3)$ une trivialisation de $\epsilon^1 \oplus T_{\partial M'}$ représentant σ' et soit v' une section non-singulière de $T_{M'}$ représentant α' et telle que $v'|_{\partial M'} = e'_1$. On note dh la différentielle de h. Alors, $dh^{-1}(e')$ représente $h^*(\sigma') \in Spin(\partial M)$ et $dh^{-1}(v')$ représente $h^*(\alpha') \in Spin^c\left(M, h^*(\sigma')\right)$. Le lemme est donc une instance de la naturalité des obstructions. □

On se propose maintenant de calculer modulo 2 ces classes de Chern relatives. Rappelons-nous que le groupe de cobordisme Ω_1^{Spin} est isomorphe à $\mathbf{Z}_2$, son générateur étant $\mathbf{S}^1$ munie de la *Spin*-structure induite par sa structure de groupe de Lie (voir [**Ki**, p. 35, 36]). Pour une surface fermée S, Johnson décrit dans [**J1**] une bijection canonique et naturelle :

$$Spin(S) \xrightarrow[\simeq]{q} Quad(\bullet_S)$$

entre les structures spinorielles de S et les formes quadratiques associées à sa forme d'intersection modulo 2. Pour $\sigma \in Spin(S)$, la forme quadratique $H_1(S; \mathbf{Z}_2) \xrightarrow{q_\sigma} \mathbf{Z}_2$ se définit comme suit. Si γ est une courbe simple fermée orientée sur S, on pose alors :

$$q_\sigma([\gamma]) = [(\gamma, \sigma|_\gamma)] \in \Omega_1^{Spin} \simeq \mathbf{Z}_2.$$

LEMME 2.33. *Soient* M *un* 3*-variété à bord,* $\sigma \in Spin(\partial M)$ *et* $\alpha \in Spin^c(M, \sigma)$. *Alors :*

$$\forall y \in H_2(M, \partial M), \quad < c(\alpha), y >= q_\sigma\left(\partial_*(y)\right) \ mod\ 2,$$

où $< ., . >$ *désigne l'évaluation de Kronecker, et où* $H_2(M, \partial M) \xrightarrow{\partial_*} H_1(\partial M)$ *est le connectant de la paire* $(M, \partial M)$.

Démonstration. La réduction modulo 2 de $c(\alpha)$ vaut :

$$w_2(M,\sigma) \in H^2(M,\partial M;\mathbf{Z}_2),$$

l'obstruction pour étendre σ à toute la variété M. Soit Σ une surface connexe immergée dans M telle que $\partial\Sigma = \partial M \cap \Sigma$, $\partial\Sigma$ n'a pas de singularité et Σ représente la réduction modulo 2 de y. Alors, $< c(\alpha), y > = < w_2(M,\sigma), y >$ mod 2 est égale à $< w_2\left(\Sigma, \sigma|_{\partial\Sigma}\right), [\Sigma] >$ modulo 2, donc est l'obstruction pour étendre la *Spin*-structure $\sigma|_{\partial\Sigma}$ à toute la surface Σ. Puisque la surface Σ est connexe, cette dernière est aussi la classe de $(\partial\Sigma, \sigma|_{\partial\Sigma})$ dans Ω_1^{Spin}. Nous avons ainsi : $< c(\alpha), y > = q_\sigma([\partial\Sigma]) = q_\sigma(\partial_*(y))$ modulo 2. □

Exemple 2.34. Supposons que M est une 3-variété de bord une réunion disjointe de tores. Notons $\rho_0 \in Parall\left((\mathbf{T}^2)_{st}\right)$ cette parallélisation préférée du tore correspondant à sa *Spin*-structure préférée σ_0 (définie dans l'Exemple 2.16). Alors, les structures d'Euler sur M relatives à $\dot\cup\rho_0$, telles qu'introduites dans la Définition 2.27, correspondent aux structures d'Euler relatives définies par Turaev dans [**T4**, §5.1] ou dans [**T8**, §1.1]. En particulier, le Lemme 2.33 est une généralisation de [**T8**, Lemme 1.3].

2.3. Recollement de structures. Comme annoncé, nous proposons maintenant des techniques de recollement pour les *Spin*-structures et les *Spin*^{c}-structures.

Dans tout ce paragraphe, nous nous donnons deux n-variétés M_1 et M_2 de bord décomposé :

$$\partial M_i = S_i \,\dot\cup\, T_i$$

où $S_i \neq \varnothing$. Nous considérons aussi un difféomorphisme $-S_2 \xrightarrow{f} S_1$, qui permet de construire la n-variété :

$$M = M_1 \cup_f M_2,$$

de bord (peut-être vide) :

$$\partial M = T_1 \,\dot\cup\, T_2.$$

L'inclusion de M_i dans M sera notée j_i.

2.3.1. *Recollement de Spin^c-structures.*

Lemme 2.35. *Soient pour $i = 1, 2$, des structures rigides $s_i \dot\cup t_i \in Spin_r^c(\partial M_i)$ telles que $f^*(s_1) = -s_2$. On suppose aussi que les obstructions relatives $w(M_i, s_i\dot\cup t_i)$ sont nulles. Alors, l'obstruction $w(M, t_1\dot\cup t_2)$ s'annule et il existe une application de recollement :*

$$Spin^c(M_1, s_1\dot\cup t_1) \times Spin^c(M_2, s_2\dot\cup t_2) \xrightarrow{\cup_f} Spin^c(M, t_1\dot\cup t_2),$$

qui est affine au-dessus de :

$$\begin{tikzcd}
H^2(M_1,\partial M_1) \oplus H^2(M_2,\partial M_2) \arrow[r, dashed] \arrow[d, "P^{-1}\times P^{-1}"'] & H^2(M,\partial M) \\
H_{n-2}(M_1) \oplus H_{n-2}(M_2) \arrow[r, "j_{1,*}\oplus j_{2,*}"] & H_{n-2}(M) \arrow[u, "P"']
\end{tikzcd}$$

où la lettre P symbolise un isomorphisme de dualité de Poincaré.

Démonstration. Soit $\alpha_i \in Spin^c(M_i, s_i\dot\cup t_i)$ représentée par la structure rigide a_i. Alors a_1 et a_2 peuvent être recollées grâce à f : on obtient une *Spin*^{c}-structure rigide sur M dont la classe d'homotopie ne dépend pas des choix de a_1 et a_2 dans leurs classes d'homotopie respectives. On la note :

$$\alpha_1 \cup_f \alpha_2 \in Spin^c(M, t_1\dot\cup t_2).$$

Montrons que l'application $\cup_f$, ainsi définie, est effectivement affine.

Les variétés M_1 et M_2 étant lisses, elles sont en particulier triangulables : soient $\mathcal{C}_1$ et $\mathcal{C}_2$ des triangulations de respectivement M_1 et M_2, telles que $\mathcal{C}_1|_{S_1}$ et $\mathcal{C}_2|_{S_2}$ se correspondent par f. Pour $i = 1$ ou 2, on note $\mathcal{C}_i^*$ la décomposition cellulaire de M_i duale de $\mathcal{C}_i$.
On forme d'une part la *réunion* $\mathcal{C}$ des triangulations $\mathcal{C}_1$ et $\mathcal{C}_2$: un simplex de $\mathcal{C}$ est un simplex de $\mathcal{C}_i$, les simplex de S_1 étant identifiés avec ceux de S_2 par f. On constitue d'autre part le *recollement* $\mathcal{C}^*$ des décompositions cellulaires $\mathcal{C}_1^*$ et $\mathcal{C}_2^*$: une cellule de $\mathcal{C}^*$ est une cellule de $\mathcal{C}_i^*$ qui ne touche pas S_i, ou bien est le recollement par f d'une cellule de $\mathcal{C}_1^*$ avec une cellule de $\mathcal{C}_2^*$ le long de faces contenues dans S_i. Alors, $\mathcal{C}$ est une triangulation de M et $\mathcal{C}^*$ est sa décomposition cellulaire duale. La première nous servira à calculer la cohomologie et la seconde servira au calcul de l'homologie.

Soient $\alpha_i, \alpha_i' \in Spin^c(M_i, s_i\dot\cup t_i)$ $(i = 1, 2)$, $\alpha := \alpha_1 \cup_f \alpha_2$ et $\alpha' := \alpha_1' \cup_f \alpha_2'$. On veut montrer cette égalité :

$$j_{1,*}P^{-1}\left(\frac{\alpha_1}{\alpha_1'}\right) + j_{2,*}P^{-1}\left(\frac{\alpha_2}{\alpha_2'}\right) = P^{-1}\left(\frac{\alpha}{\alpha'}\right) \in H_{n-2}(M). \tag{2.3}$$

Soient $a_i, a_i' \in Spin_r^c(M_i)$ des représentants respectifs de α_i et α_i' qui coïncident sur le 1-squelette de $\mathcal{C}_i$ (ainsi que, bien sûr, sur ∂M_i). Nous avons fixé un morphisme de fibrés $\tau_{M_i} \longrightarrow \gamma_{SO}$, et, comme dans la preuve de la Proposition 2.6, nous identifions la structure rigide a_i à un relèvement $M_i \longrightarrow BSpin^c$ de l'application sous-jacente pour les espaces bases $M_i \longrightarrow BSO$. Alors $\alpha_i/\alpha_i' \in H^2(M_i, \partial M_i)$ est la classe du 2-cocycle qui assigne à chaque 2-cellule e_k^i de $\mathcal{C}_i$ en-dehors de ∂M_i, cet élément z_k^i de $\pi_2(BU(1)) \simeq \pi_2(K(\mathbf{Z}, 2)) \simeq \mathbf{Z}$ obtenu par recollement de $a_i|_{e_k^i}$ et $a_i'|_{e_k^i}$ le long de ∂e_k^i. Alors, $P^{-1}(\alpha_i/\alpha_i') = [\sum_k z_k^i \cdot e_k^{*,i}]$ où $e_k^{*,i}$ est la $(n-2)$-cellule duale de e_k^i.
De plus, $a := a_1 \cup_f a_2$ et $a' := a_1' \cup_f a_2'$ représentent respectivement α et α'. De façon similaire, en utilisant ces structures rigides, on peut décrire un 2-cocycle représentant α/α'. Ce 2-cocycle envoie chaque 2-cellule de $\mathcal{C}_1 \cup_f \mathcal{C}_2$ contenue dans $S_1 = -S_2$ sur $0 \in \mathbf{Z}$ si bien que $P^{-1}(\alpha/\alpha')$ est représenté par $\sum_k z_k^1 \cdot e_k^{*,1} + \sum_k z_k^2 \cdot e_k^{*,2}$. $\square$

En dimension 3, nous travaillerons avec les $Spin^c$-structures relatives à des structures spinorielles (telles qu'introduites dans la Définition 2.15). Voici le lemme de recollement que nous pratiquerons pour les $Spin^c$-structures :

COROLLAIRE 2.36. *Supposons la dimension n égale à 3. Soient pour $i = 1, 2$, des structures spinorielles $\sigma_i\dot\cup\tau_i \in Spin(\partial M_i)$ telles que $f^*(\sigma_1) = -\sigma_2$. Il existe alors une application de recollement :*

$$Spin^c(M_1, \sigma_1\dot\cup\tau_1) \times Spin^c(M_2, \sigma_2\dot\cup\tau_2) \xrightarrow{\cup_f} Spin^c(M, \tau_1\dot\cup\tau_2),$$

qui est affine au-dessus de :

$$\begin{tikzcd}
H^2(M_1, \partial M_1) \oplus H^2(M_2, \partial M_2) \arrow[r, dashed] \arrow[d, "P^{-1} \times P^{-1}"'] & H^2(M, \partial M) \\
H_1(M_1) \oplus H_1(M_2) \arrow[r, "j_{1,*} \oplus j_{2,*}"] & H_1(M) \arrow[u, "P"']
\end{tikzcd}$$

En outre, pour $\alpha_i \in Spin^c(M, \sigma_i\dot\cup\tau_i)$ $(i = 1, 2)$, l'égalité suivante entre classes de Chern est satisfaite :

$$P^{-1}c(\alpha_1 \cup_f \alpha_2) = j_{1,*}P^{-1}c(\alpha_1) + j_{2,*}P^{-1}c(\alpha_2) \in H_1(M).$$

Démonstration. D'après le Th. 2.14, les obstructions relatives s'annulent dans ce contexte : la première assertion du corollaire est alors une application directe du Lemme 2.35 et de la définition des $Spin^c$-structures relatives à des $Spin$-structures. La deuxième assertion est encore un calcul de recollement d'obstructions sur des variétés orientées (profitant de la dualité de Poincaré) : la preuve est similaire à celle du Lemme 2.35. □

2.3.2. *Recollement de Spin-structures.* Le lemme suivant se prouve comme le Lemme 2.35.

Lemme 2.37. *Soient pour $i = 1, 2$, des structures rigides $s_i \dot\cup t_i \in Spin_r(\partial M_i)$ telles que $f^*(s_1) = -s_2$. On suppose aussi que les obstructions relatives $w_2(M_i, s_i\dot\cup t_i) \in H^2(M_i, \partial M_i; \mathbf{Z}_2)$ sont nulles. Alors, l'obstruction $w_2(M, t_1\dot\cup t_2)$ s'annule et il existe une application de recollement :*

$$Spin(M_1, s_1\dot\cup t_1) \times Spin(M_2, s_2\dot\cup t_2) \xrightarrow{\cup_f} Spin(M, t_1\dot\cup t_2),$$

qui est affine au-dessus de :

$$\begin{tikzcd}
H^1(M_1, \partial M_1; \mathbf{Z}_2) \oplus H^1(M_2, \partial M_2; \mathbf{Z}_2) \arrow[r, dashed] \arrow[d, "P^{-1} \times P^{-1}"'] & H^1(M, \partial M; \mathbf{Z}_2) \\
H_{n-1}(M_1; \mathbf{Z}_2) \oplus H_{n-1}(M_2; \mathbf{Z}_2) \arrow[r, "j_{1,*} \oplus j_{2,*}"] & H_{n-1}(M; \mathbf{Z}_2). \arrow[u, "P"']
\end{tikzcd}$$

En pratique, nous travaillerons en dimension 3 avec les $Spin$-structures relatives introduites dans la Définition 2.17.

Corollaire 2.38. *Supposons la dimension n égale à 3 et que le lieu de recollement $S_1 \cong (-S_2)$ est* connexe. *Soient pour $i = 1, 2$, des structures spinorielles $\sigma_i\dot\cup\tau_i \in Spin(\partial M_i)$ telles que $f^*(\sigma_1) = -\sigma_2$ et telles que les obstructions relatives $w_2(M_i, \sigma_i\dot\cup\tau_i)$ s'annulent. Il existe alors une unique application de recollement :*

$$Spin(M_1, \sigma_1\dot\cup\tau_1) \times Spin(M_2, \sigma_2\dot\cup\tau_2) \xrightarrow{\cup_f} Spin(M, \tau_1\dot\cup\tau_2),$$

telle que, si $\alpha_i \in Spin(M_i, \sigma_i\dot\cup\tau_i)$ et si $Spin(M) \xrightarrow{j_i^*} Spin(M_i)$ désigne l'application de restriction, alors :

$$\forall i \in \{1, 2\},\ j_i^*(\alpha_1 \cup_f \alpha_2) = \alpha_i \in Spin(M_i).$$

De plus, ce diagramme est commutatif :

$$\begin{tikzcd}
Spin(M_1, \sigma_1\dot\cup\tau_1) \times Spin(M_2, \sigma_2\dot\cup\tau_2) \arrow[r, "\cup_f"] \arrow[d, "\beta \times \beta"'] & Spin(M, \tau_1\dot\cup\tau_2) \arrow[d, "\beta"] \\
Spin^c(M_1, \sigma_1\dot\cup\tau_1) \times Spin^c(M_2, \sigma_2\dot\cup\tau_2) \arrow[r, "\cup_f"] & Spin^c(M, \tau_1\dot\cup\tau_2).
\end{tikzcd}$$

Démonstration. Soient $\alpha_1 \in Spin(M_1, \sigma_1\dot\cup\tau_1)$ et $\alpha_2 \in Spin(M_2, \sigma_2\dot\cup\tau_2)$. On choisit $a_i \in Spin_r(M_i)$ représentant α_i $(i = 1, 2)$. Par hypothèse, $f^*(a_1|_{S_1})$ est homotope à $-a_2|_{S_2}$. Quitte à homotoper a_2 dans sa classe α_2, on peut donc supposer l'égalité de structures rigides $f^*(a_1|_{S_1}) = -a_2|_{S_2}$, et construire grâce au Lemme 2.37 la structure :

$$[a_1] \cup_f [a_2] \in Spin\left(M, (a_1|_{T_1}) \dot\cup (a_2|_{T_2})\right),$$

induisant une certaine structure $\alpha \in Spin(M, \tau_1\dot\cup\tau_2)$. Notons que, par construction, $\alpha \in Spin(M)$ satisfait $j_i^*(\alpha) = \alpha_i \in Spin(M_i)$.

D'autres choix de structures rigides $a'_i \in Spin_r(M_i)$ représentant α_i $(i = 1,2)$ conduiront par ce procédé à une structure spinorielle $\alpha' \in Spin(M, \tau_1 \dot{\cup} \tau_2)$ vérifiant elle-aussi $j_i^*(\alpha') = \alpha_i$. Or, l'application de restrictions :

$$Spin(M) \xrightarrow{(j_1^*, j_2^*)} Spin(M_1) \times Spin(M_2),$$

est injective car elle est affine au-dessus de :

$$H^1(M; \mathbf{Z}_2) \xrightarrow{(j_1^*, j_2^*)} H^1(M_1; \mathbf{Z}_2) \oplus H^1(M_2; \mathbf{Z}_2)$$

qui est elle-même injective puisque $S_1 \cong -S_2$ est connexe. D'où, $\alpha' = \alpha$ ce qui nous autorise à poser $\alpha_1 \cup_f \alpha_2 := \alpha$. □

3. Raffinements spinoriels de la théorie de Goussarov-Habiro

Grâce aux techniques de recollement de structures introduites dans la section précédente, nous allons pouvoir raffiner la théorie de Goussarov-Habiro.

Conventions 3.1. De nouveau, nous nous restreignons aux 3-variétés, que nous supposerons lisses compactes orientées et éventuellement à bord.

3.1. Twists. On se place dans la situation de recollement étudiée au §2.3. On considère donc deux 3-variétés M_1 et M_2 de bord décomposé : $\partial M_i = S_i \dot{\cup} T_i$, où $S_i \neq \varnothing$ et où T_i est éventuellement vide. Nous considérons aussi un difféomorphisme positif $-S_2 \xrightarrow{f} S_1$, qui nous permet de former la 3-variété :

$$M = M_1 \cup_f M_2,$$

de bord $\partial M = T_1 \dot{\cup} T_2$, peut-être vide. L'inclusion de M_i dans M est de nouveau notée j_i. *On suppose que le lieu de recollement $S_1 \cong -S_2$ est connexe.*

La 3-variété M peut être modifiée en la twistant le long de la surface plongée S_2 par un élément de :

$$\mathcal{T}(S_2) = \frac{\left\{ S_2 \xrightarrow[\cong]{h} S_2 \ : \ h \text{ agit trivialement en homologie} \right\}}{\text{isotopie}},$$

groupe de Torelli de la surface S_2. Précisément, *la 3-variété obtenue en twistant M le long de la surface S_2 par $h \in \mathcal{T}(S_2)$* est définie par :

$$M' = M_1 \cup_{f \circ h} M_2. \tag{3.1}$$

Notons que $\partial M'$ s'identifie canoniquement à ∂M : $\partial M' = T_1 \dot{\cup} T_2$. Nous noterons $M_i \overset{j'_i}{\hookrightarrow} M'$ l'inclusion de M_i dans M' définie par (3.1). Par un argument du type Mayer-Vietoris, il exite un isomorphisme d'homologie :

$$H_1(M) \xrightarrow{\Phi} H_1(M')$$

défini sans ambiguïté par ce diagramme :

$$\begin{tikzcd}
 & H_1(M) \arrow[dd, "\simeq"', "\Phi"] & \\
H_1(M_1) \arrow[ur, "j_{1,*}"] \arrow[dr, "j'_{1,*}"'] & & H_1(M_2) \arrow[ul, "j_{2,*}"'] \arrow[dl, "j'_{2,*}"] \\
 & H_1(M') &
\end{tikzcd}$$

La proposition suivante affirme qu'un tel twist s'accompagne d'une correspondance canonique entre les $Spin^c$-structures de la variété initiale et celles de la variété résultante.

PROPOSITION 3.2. *a) Si $\partial M = \varnothing$, il existe alors une bijection canonique :*

$$Spin^c(M) \underset{\simeq}{\xrightarrow{\Omega}} Spin^c(M'),$$

affine au-dessus de $H^2(M) \xrightarrow{P\Phi P^{-1}} H^2(M')$ et telle que ce diagramme commute :

$$\begin{tikzcd} Spin^c(M) \arrow[r, "\Omega"] \arrow[d, "c"'] & Spin^c(M') \arrow[d, "c"] \\ H^2(M) \arrow[r, "P\Phi P^{-1}"'] & H^2(M'). \end{tikzcd}$$

b) Si $\partial M \neq \varnothing$, et si $\gamma \in Spin(\partial M)$ correspond à $\gamma' \in Spin(\partial M')$, il existe alors une bijection canonique :

$$Spin^c(M,\gamma) \underset{\simeq}{\xrightarrow{\Omega}} Spin^c(M',\gamma')$$

affine au-dessus de $H^2(M,\partial M) \underset{\simeq}{\xrightarrow{P\Phi P^{-1}}} H^2(M',\partial M')$ et telle que ce diagramme commute :

$$\begin{tikzcd} Spin^c(M,\gamma) \arrow[r, "\Omega"] \arrow[d, "c"'] & Spin^c(M',\gamma') \arrow[d, "c"] \\ H^2(M,\partial M) \arrow[r, "P\Phi P^{-1}"'] & H^2(M',\partial M'). \end{tikzcd}$$

DÉMONSTRATION. Nous ne démontrons la proposition que dans le cas fermé, la preuve dans le cas à bord étant en tous points similaire.

Pour $\alpha \in Spin^c(M)$, $\Omega(\alpha)$ se définit comme suit. *Choisissons* une structure spinorielle σ_2 sur S_2 et notons aussi $\sigma_1 = f_*(-\sigma_2) \in Spin(S_1)$. Comme h induit l'identité au niveau de $H_1(S_2;\mathbf{Z}_2)$, h agit trivialement[5] sur $Spin(S_2)$: donc $(fh)_*(-\sigma_2) = \sigma_1$. D'après le Corollaire 2.36, nous disposons donc de deux applications de recollement :

$$Spin^c(M_1,\sigma_1) \times Spin^c(M_2,\sigma_2) \xrightarrow{\cup_f} Spin^c(M),$$

$$\text{et :}\quad Spin^c(M_1,\sigma_1) \times Spin^c(M_2,\sigma_2) \xrightarrow{\cup_{f\circ h}} Spin^c(M'),$$

qui sont respectivement affines, *via* les dualités de Poincaré, au-dessus de $j_{1,*} \oplus j_{2,*}$ et $j'_{1,*} \oplus j'_{2,*}$. Puisque S_2 est connexe, l'application $\cup_f$ est surjective. *Choisissons* $\alpha_1 \in Spin^c(M_1,\sigma_1)$ et $\alpha_2 \in Spin^c(M_2,\sigma_2)$, telles que $\alpha = \alpha_1 \cup_f \alpha_2$. On définit alors :

$$\alpha' = \alpha_1 \cup_{fh} \alpha_2 \in Spin^c(M').$$

[5]Ceci découle par exemple de la bijection naturelle entre $Spin(S_2)$ et $Quad\left(\bullet_{S_2}\right)$, telle que rappelée au §2.2.5.

et nous posons $\Omega(\alpha) := \alpha' \in Spin^c(M')$.

Vérifions maintenant que la procédure précédente définit sans équivoque $\Omega(\alpha)$. Envisageons d'autres *choix* $\tilde{\sigma}_2$, $\tilde{\alpha}_1$ et $\tilde{\alpha}_2$ de respectivement σ_2, α_1 et α_2, conduisant à $\tilde{\alpha}' = \tilde{\alpha}_1 \cup_{fh} \tilde{\alpha}_2$; nous prétendons que $\alpha' = \tilde{\alpha}'$.

Plaçons-nous d'abord dans ce cas particulier où $\tilde{\sigma}_2 = \sigma_2 \in Spin(S_2)$. Alors, comme $\alpha_1 \cup_f \alpha_2 = \alpha = \tilde{\alpha}_1 \cup_f \tilde{\alpha}_2$, on a :

$$j_{1,*}P^{-1}\begin{pmatrix}\alpha_1\\ \tilde{\alpha}_1\end{pmatrix} + j_{2,*}P^{-1}\begin{pmatrix}\alpha_2\\ \tilde{\alpha}_2\end{pmatrix} = P^{-1}\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} = 0 \in H_1(M).$$

En appliquant Φ à cette dernière égalité, on obtient cette identité :

$$j'_{1,*}P^{-1}\begin{pmatrix}\alpha_1\\ \tilde{\alpha}_1\end{pmatrix} + j'_{2,*}P^{-1}\begin{pmatrix}\alpha_2\\ \tilde{\alpha}_2\end{pmatrix} = 0 \in H_1(M'),$$

dont le terme de gauche égale $P^{-1}(\alpha'/\tilde{\alpha}')$ vu que $\alpha' = \alpha_1 \cup_{fh} \alpha_2$ et que $\tilde{\alpha}' = \tilde{\alpha}_1 \cup_{fh} \tilde{\alpha}_2$. On en déduit que $\alpha' = \tilde{\alpha}'$.

Traitons maintenant le cas général. Piochons une structure $s_2 \in Spin_r(S_2)$ dans la classe σ_2 et soit $s_1 = f_*(-s_2) \in Spin_r(S_1)$, qui donc représente σ_1. Soient aussi pour $i = 1$ et 2, une structure $a_i \in Spin^c_r(M_i)$ représentant α_i et telle que $a_i|_{S_i} = \beta(s_i)$. Alors :

$$\alpha = [a_1 \cup_f a_2] \in Spin^c(M), \tag{3.2}$$

où $\cup_f$ désigne ici la version rigide de l'application de recollement (celle du Lemme 2.35). Soit aussi une homotopie entre $h^*(s_2)$ et s_2 induisant une *Spin*-structure rigide notée $U \in Spin_r(S_2 \times I)$. Alors,

$$\alpha' = [a_1 \cup_{fh} (\beta(U) \cup a_2)] \in Spin^c(M'). \tag{3.3}$$

Soit $\tilde{s}_2 \in Spin_r(S_2)$ représentant $\tilde{\sigma}_2$; on pose alors $\tilde{s}_1 = f_*(-\tilde{s}_2) \in Spin_r(S_1)$: elle représente $\tilde{\sigma}_1 := f_*(-\tilde{\sigma}_2) \in Spin(S_1)$. Quoique σ_2 et $\tilde{\sigma}_2$ puissent être différentes, on a toujours : $\beta(\sigma_2) = \beta(\tilde{\sigma}_2) \in Spin^c(S_2)$. Il existe donc une homotopie entre $\beta(\tilde{s}_2)$ et $\beta(s_2)$ induisant une $Spin^c$-structure rigide notée $H \in Spin^c_r(S_2 \times I)$. Alors, $f_*(-H) \in Spin^c_r(S_1 \times I)$ est une homotopie entre $\beta(s_1)$ et $\beta(\tilde{s}_1)$. Maintenant, la $Spin^c$-structure rigide sur M :

$$(a_1 \cup f_*(-H)) \cup_f (H \cup a_2)$$

peut être homotopée dans M, grâce au double collier de $j_2(S_2)$, à $a_1 \cup_f a_2$. L'application de recollement :

$$Spin^c(M_1, \tilde{\sigma}_1) \times Spin^c(M_2, \tilde{\sigma}_2) \xrightarrow{\cup_f} Spin^c(M),$$

envoie donc, d'après (3.2), le couple $([a_1 \cup f_*(-H)], [H \cup a_2])$ sur α. D'après l'étude du cas particulier précédent, quelqu'aient pu être les choix de $\tilde{\alpha}_1$ et $\tilde{\alpha}_2$, on a donc :

$$\tilde{\alpha}' = [a_1 \cup f_*(-H)] \cup_{fh} [H \cup a_2].$$

Ainsi, pour une homotopie entre $h^*(\tilde{s}_2)$ et $\tilde{s}_2$ induisant $V \in Spin_r(S_2 \times I)$, nous obtenons :

$$\tilde{\alpha}' = [(a_1 \cup f_*(-H)) \cup_{fh} (\beta(V) \cup H \cup a_2)]. \tag{3.4}$$

De même que précedemment pour M, la $Spin^c$-structure rigide sur M' :

$$(a_1 \cup f_*(-H)) \cup_{fh} (h^*(H) \cup \beta(U) \cup a_2)$$

peut-être homotopée dans M', grâce au double collier de $j'_2(S_2)$, à $a_1 \cup_{fh} (\beta(U) \cup a_2)$. D'après (3.3), nous avons donc :

$$\alpha' = [(a_1 \cup f_*(-H)) \cup_{fh} (h^*(H) \cup \beta(U) \cup a_2)]. \tag{3.5}$$

D'après (3.4) et (3.5), pour prouver que $\tilde{\alpha}' = \alpha'$, il nous suffit de montrer cette égalité de $Spin^c$-structures relatives à des structures spinorielles :

$$[\beta(V) \cup H] = [h^*(H) \cup \beta(U)] \in Spin^c\left(S_2 \times I, \tilde{\sigma}_2 \dot{\cup} (-\sigma_2)\right). \tag{3.6}$$

Puisque $H^2\left(S_2 \times I, \partial\left(S_2 \times I\right)\right)$ n'a pas de 2-torsion, ces $Spin^c$-structures relatives sont déterminées par leurs classes de Chern relatives. Ayant :

$$c\left([\beta(U)]\right) = 0 \quad \text{et :} \quad c\left([\beta(V)]\right) = 0,$$

d'après la Remarque 2.31, et le Lemme 2.32 impliquant : $c\left([h^*(H)]\right) = h^*\left(c\left([H]\right)\right) = c\left([H]\right)$, l'égalité (3.6) est satisfaite d'après la dernière assertion du Corollaire 2.36. On conclut donc à la bonne définition de l'application Ω.

On vérifie maintenant que l'application Ω est affine. Soient pour cela $\alpha \in Spin^c(M)$ et $x \in H_1(M)$. Ayant choisie une $Spin$-structure σ_2 sur S_2 et notant $\sigma_1 = f_*(-\sigma_2)$, on écrit α sous la forme :

$$\alpha = \alpha_1 \cup_f \alpha_2 \in Spin^c(M),$$

où $\alpha_i \in Spin^c(M,\sigma_i)$. On peut aussi écrire x sous la forme $x = j_{1,*}(x_1) + j_{2,*}(x_2)$ où $x_i \in H_1(M_i)$. Alors, $P(x) \cdot \alpha = \left(P(x_1) \cdot \alpha_1\right) \cup_f \left(P(x_2) \cdot \alpha_2\right)$, d'où :

$$\begin{aligned}
\Omega\left(P(x) \cdot \alpha\right) &= \left(P(x_1) \cdot \alpha_1\right) \cup_{fh} \left(P(x_2) \cdot \alpha_2\right) \\
&= P\left(j'_{1,*}(x_1) + j'_{2,*}(x_2)\right) \cdot \left(\alpha_1 \cup_{fh} \alpha_2\right) \\
&= P\left(\Phi j_{1,*}(x_1) + \Phi j_{2,*}(x_2)\right) \cdot \Omega(\alpha) \\
&= P\Phi(x) \cdot \Omega(\alpha).
\end{aligned}$$

Enfin, avec ces mêmes notations :

$$\begin{aligned}
P^{-1}c\left(\Omega(\alpha)\right) &= P^{-1}c\left(\alpha_1 \cup_{fh} \alpha_2\right) \\
&= j'_{1,*}\left(P^{-1}c(\alpha_1)\right) + j'_{2,*}\left(P^{-1}c(\alpha_2)\right) \\
&= \Phi\left(j_{1,*}\left(P^{-1}c(\alpha_1)\right) + j_{2,*}\left(P^{-1}c(\alpha_2)\right)\right) \\
&= \Phi\left(P^{-1}\left(c(\alpha)\right)\right) \in H_1(M').
\end{aligned}$$

$\square$

Passons maintenant aux structures spinorielles. Les évaluations de Kronecker permettent les identifications :

$$H^1(M;\mathbf{Z}_2) \simeq Hom(H_1(M),\mathbf{Z}_2) \quad \text{et :} \quad H^1(M';\mathbf{Z}_2) \simeq Hom(H_1(M'),\mathbf{Z}_2),$$

si bien que $Hom(\Phi,\mathbf{Z}_2)$ définit un isomorphisme :

$$H^1(M';\mathbf{Z}_2) \xrightarrow{\Phi^{(2)}} H^1(M;\mathbf{Z}_2).$$

Proposition 3.3. *Il existe alors une bijection canonique :*

$$Spin(M) \underset{\simeq}{\xrightarrow{\Theta}} Spin(M')$$

qui est affine au-dessus de $H^1(M;\mathbf{Z}_2) \xrightarrow{\left(\Phi^{(2)}\right)^{-1}} H^1(M';\mathbf{Z}_2)$. *En outre :*
a) si $\partial M = \varnothing$, *le diagramme suivant commute :*

$$\begin{tikzcd}
Spin(M) \arrow[r, "\Theta", "\simeq"'] \arrow[d, "\beta"'] & Spin(M') \arrow[d, "\beta"] \\
Spin^c(M) \arrow[r, "\simeq", "\Omega"'] & Spin^c(M');
\end{tikzcd}$$

b) si $\partial M \neq \varnothing$ et si $\gamma \in Spin(\partial M)$ est telle que $w_2(M,\gamma) = 0$ et correspond à $\gamma' \in Spin(\partial M')$, alors $w_2(M',\gamma') = 0$, $\Theta\left(Spin(M,\gamma)\right) = Spin(M',\gamma')$ et le diagramme suivant commute :

$$\begin{tikzcd} Spin(M,\gamma) \arrow[r, "\Theta|", "\simeq"'] \arrow[d, "\beta"'] & Spin(M',\gamma') \arrow[d, "\beta"] \\ Spin^c(M,\gamma) \arrow[r, "\simeq", "\Omega"'] & Spin^c(M',\gamma'). \end{tikzcd}$$

Démonstration. Supposons ∂M non-vide : la preuve dans le cas $\partial M = \varnothing$, par ailleurs plus courte, s'en déduit aisément.

Soient $\gamma \in Spin(\partial M)$ telle que $w_2(M,\gamma) = 0$ et $\alpha \in Spin(M,\gamma)$. On note $\sigma_i \in Spin(S_i)$, $\tau_i \in Spin(T_i)$ et $\alpha_i \in Spin(M_i, \sigma_i \dot{\cup} \tau_i)$ les restrictions de α $(i = 1,2)$. D'après le Corollaire 2.38, nous avons donc :

$$\alpha = \alpha_1 \cup_f \alpha_2 \in Spin(M,\gamma). \tag{3.7}$$

Comme h agit trivialement en homologie, $(fh)_*(-\sigma_2) = \sigma_1$. On peut donc poser :

$$\alpha' = \alpha_1 \cup_{fh} \alpha_2 \in Spin(M',\gamma'). \tag{3.8}$$

et définir $\Theta(\alpha) := \alpha'$. Le fait que $\beta\left(\Theta(\alpha)\right) = \Omega\left(\beta(\alpha)\right)$ découle de (3.7), (3.8), de la définition de l'application Ω et de la dernière assertion du Corollaire 2.38.

Montrons que Θ est affine. Soit $y \in H^1(M;\mathbf{Z}_2)$. Puisque $j_i^*(y \cdot \alpha) = j_i^*(y) \cdot \alpha_i \in Spin(M_i)$ $(i = 1,2)$, nous avons :

$$\begin{array}{rcll} \Theta(y \cdot \alpha) & = & (j_1^*(y) \cdot \alpha_1) \cup_{fh} (j_2^*(y) \cdot \alpha_2) & \\ & = & P\left(j'_{1,*}P^{-1}j_1^*(y) + j'_{2,*}P^{-1}j_2^*(y)\right) \cdot (\alpha_1 \cup_{fh} \alpha_2) & \\ & = & P\left(j'_{1,*}P^{-1}j_1^*(y) + j'_{2,*}P^{-1}j_2^*(y)\right) \cdot \Theta(\alpha) & \in Spin(M'). \end{array}$$

Il suffit donc de montrer que $P\left(j'_{1,*}P^{-1}j_1^*(y) + j'_{2,*}P^{-1}j_2^*(y)\right) = (\Phi^{(2)})^{-1}(y) \in H^1(M';\mathbf{Z}_2)$. Or, pour tout $x \in H_1(M)$ tel que $x = j_{1,*}(x_1) + j_{2,*}(x_2)$ où $x_i \in H_1(M_i)$, nous avons :

$$\begin{array}{cl} & < \Phi^{(2)} P\left(j'_{1,*}P^{-1}j_1^*(y) + j'_{2,*}P^{-1}j_2^*(y)\right), x > \\ = & < P\left(j'_{1,*}P^{-1}j_1^*(y) + j'_{2,*}P^{-1}j_2^*(y)\right), \Phi(x) > \\ = & < P\left(j'_{1,*}P^{-1}j_1^*(y) + j'_{2,*}P^{-1}j_2^*(y)\right), j'_{1,*}(x_1) + j'_{2,*}(x_2) > \\ = & \left(j'_{1,*}P^{-1}j_1^*(y) + j'_{2,*}P^{-1}j_2^*(y)\right) \bullet_{M'} \left(j'_{1,*}(x_1) + j'_{2,*}(x_2)\right) \\ = & \left(P^{-1}j_1^*(y)\right) \bullet_{M_1} x_1 + \left(P^{-1}j_2^*(y)\right) \bullet_{M_2} x_2 \\ = & < j_1^*(y), x_1 > + < j_2^*(y), x_2 > \\ = & < y, j_{1,*}(x_1) + j_{2,*}(x_2) > \\ = & < y, x > \end{array}$$

ce qui conclut la preuve de la proposition. □

Remarque 3.4. Dans le cas où $\partial M = \varnothing$, on déduit des propositions numérotées 3.2 et 3.3 que le diagramme suivant commute :

$$\begin{tikzcd} H^1(M;\mathbf{Z}_2) \arrow[d, "\beta"'] & H^1(M';\mathbf{Z}_2) \arrow[l, "\Phi^{(2)}"'] \arrow[d, "\beta"] \\ H^2(M) \arrow[r, "P\Phi P^{-1}"] & H^2(M') \end{tikzcd}$$

ce qui peut se vérifier directement. En effet, notons $H^1(M;\mathbf{Z}_2) \xrightarrow{D} T_2H_1(M)$ la composée $P^{-1}\beta$ à valeur dans le sous-groupe de 2-torsion de $H_1(M)$. L'homomorphisme D est caractérisé par la propriété suivante :

$$\forall y \in H^1(M;\mathbf{Z}_2),\ \forall x \in TH_1(M), \quad \lambda_M(D(y),x) = \frac{1}{2}\cdot < y, x >,$$

où λ_M désigne la forme d'enlacement de M et où $\frac{1}{2}\cdot$ est l'inclusion usuelle de $\mathbf{Z}_2$ dans $\mathbf{Q}/\mathbf{Z}$. L'isomorphisme Φ, restreint aux sous-groupes de torsion, vérifie : $\lambda_{M'} \circ (\Phi| \otimes \Phi|) = \lambda_M$. Prenons $y \in H^1(M';\mathbf{Z}_2)$. On a alors pour tout $y' \in H^1(M';\mathbf{Z}_2)$ et pour tout $x' \in TH_1(M')$:

$$\begin{array}{rcl} \lambda_{M'}\left(\Phi D \Phi^{(2)}(y'), x'\right) & = & \lambda_M\left(D\Phi^{(2)}(y'), \Phi^{-1}(x')\right) \\ & = & \frac{1}{2} < \Phi^{(2)}(y'), \Phi^{-1}(x') > \\ & = & \frac{1}{2} < y', x' > \\ & = & \lambda_{M'}(D(y'), x') \end{array}$$

d'où : $\Phi D \Phi^{(2)} = D$, ce qui démontre la commutativité du précédent diagramme.

3.2. Calcul de clovers spinoriel. Nous sommes maintenant en mesure de montrer que les chirurgies le long de clovers font sens pour les variétés munies de *Spin*-structures ou de *Spin*^{c}-structures.

3.2.1. *Y_k-chirurgies spinorielles complexes.* Soient une 3-variété M et un clover G dans M. On autorise M à avoir du bord, auquel cas nous fixons une *Spin*-structure σ sur ∂M et nous notons encore $\sigma \in Spin\,(\partial M_G)$ la structure correspondante par l'identification canonique $\partial M \cong \partial M_G$.

LEMME 3.5. *La chirurgie le long de G induit une bijection canonique :*

$$Spin^c(M) \underset{\simeq}{\xrightarrow{\Omega_G}} Spin^c(M_G) \quad ou : \quad Spin^c(M,\sigma) \underset{\simeq}{\xrightarrow{\Omega_G}} Spin^c(M_G,\sigma),$$

suivant que M est fermée ou à bord. Cette correspondance est notée : $\alpha \longmapsto \alpha_G$. En outre, l'application Ω_G est affine au-dessus de $P\Phi_G P^{-1}$, où Φ_G est l'isomorphisme d'homologie qui est défini sans équivoque par le diagramme commutatif suivant :

$$\begin{tikzcd} & H_1(M) \arrow[dd, "\Phi_G", "\simeq"'] \\ H_1(M \setminus N(G)) \arrow[ur, "k_*"] \arrow[dr, "k'_*"'] & \\ & H_1(M_G). \end{tikzcd}$$

Ici, $N(G)$ désigne un voisinage régulier de G dans M, $M \setminus N(G) \overset{k}{\hookrightarrow} M$ et $M \setminus N(G) \overset{k'}{\hookrightarrow} M_G$ sont les inclusions.

DEFINITION 3.6. La *Spin*^{c}-variété (M_G, α_G) est dite obtenue de (M,α) par *Y_k-chirurgie spinorielle complexe*, ou *Y_k^c-chirurgie*, le long de G.
On appelle *Y_k-équivalence spinorielle complexe*, ou *Y_k^c-équivalence*, la relation d'équivalence au sein des *Spin*^{c}-variétés de dimension 3, engendrée par les Y_k^c-chirurgies et les *Spin*^{c}-difféomorphismes.

Démonstration du Lemme 3.5. Tout d'abord, nous envisageons le cas où G est un Y-graphe. Nous fixons un plongement positif $H_3 \overset{j}{\hookrightarrow} M$ d'image un voisinage régulier $N(G)$ de G dans M. D'après la Remarque 1.5, il existe un automorphisme h de $\Sigma_3 = \partial H_3$ qui a la propriété d'agir trivialement en homologie et d'être tel qu'il existe un difféomorphisme :

$$M_G = M \setminus int\,(N(G)) \cup_{j|_\partial} (H_3)_L \underset{\cong}{\xrightarrow{f}} M \setminus int\,(N(G)) \cup_{j|_\partial \circ h} H_3,$$

qui est l'identité sur $M \setminus int\,(N(G))$. Il y a donc une bijection :

$$Spin^c\,(M_G) \underset{\simeq}{\xrightarrow{f_*}} Spin^c\left(M \setminus int\,(N(G)) \cup_{j|_\partial \circ h} H_3\right).$$

D'après la Prop. 3.2, il y a aussi une bijection canonique :

$$Spin^c\,(M) \underset{\simeq}{\xrightarrow{\Omega}} Spin^c\left(M \setminus int\,(N(G)) \cup_{j|_\partial \circ h} H_3\right).$$

Nous posons alors $\Omega_G = f_*^{-1}\Omega$. Nous devons vérifier que l'application Ω_G ne dépend pas des différents choix intermédiaires.

Nous vérifions l'indépendance en la paire (h, f) vérifiant la propriété énoncée plus haut. Soit (h', f') un autre choix, il s'agit de prouver que le diagramme suivant commute :

$$\begin{tikzcd}
 & Spin^c\left(M \setminus int\,(N(G)) \cup_{j|_\partial \circ h} H_3\right) \arrow[dd, "g_*"] \\
Spin^c(M) \arrow[ur, "\Omega"] \arrow[dr, "\Omega'"'] & \\
 & Spin^c\left(M \setminus int\,(N(G)) \cup_{j|_\partial \circ h'} H_3\right)
\end{tikzcd}$$

où $g = f' \circ f^{-1}$. Soit $\alpha \in Spin^c(M)$. Choisissons une structure $\sigma \in Spin(\Sigma_3)$, et écrivons α sous la forme $\alpha_1 \cup j_*(\alpha_2)$ où $\alpha_1 \in Spin^c\,(M \setminus int\,(N(G))\,, -j_*(\sigma))$ et $\alpha_2 \in Spin^c(H_3, \sigma)$. Alors, $\Omega(\alpha) = \alpha_1 \cup_{j|_\partial \circ h} \alpha_2$ et $\Omega'(\alpha) = \alpha_1 \cup_{j|_\partial \circ h'} \alpha_2$. Nous avons : $g_*\,(\Omega(\alpha)) = \alpha_1 \cup_{j|_\partial \circ h'} (g|)_*\,(\alpha_2)$. Ici, $Spin^c(H_3, \sigma) \xrightarrow{(g|)_*} Spin^c(H_3, \sigma)$ est l'identité puisque $g|_{H_3}$ agit trivialement en homologie et puisque les éléments de $Spin^c(H_3, \sigma)$ sont classifiés par leurs classes de Chern relatives. Il s'en suit que : $g_*\,(\Omega(\alpha)) = \Omega'(\alpha)$.

Ensuite, nous vérifions l'indépendance en j. Soit j' une trivialisation d'un autre voisinage régulier $N'(G)$ de G dans M et soit $\left(M \xrightarrow{q_t} M\right)_{t\in[0,1]}$ une isotopie ambiante entre j et j' : $q_0 = Id_M$ et $q_1 \circ j = j'$. Le difféomorphisme q_1 induit un difféomorphisme :

$$M \setminus int\,(N(G)) \cup_{j|_\partial \circ h} H_3 \xrightarrow{q} M \setminus int\,(N'(G)) \cup_{j'|_\partial \circ h} H_3.$$

Justifier "l'indépendance en j" signifie prouver la commutativité du diagramme suivant :

$$\begin{tikzcd}
 & Spin^c\left(M \setminus int\,(N(G)) \cup_{j|_\partial \circ h} H_3\right) \arrow[dd, "q_*"] \\
Spin^c(M) \arrow[ur, "\Omega"] \arrow[dr, "\Omega'"'] & \\
 & Spin^c\left(M \setminus int\,(N'(G)) \cup_{j'|_\partial \circ h} H_3\right).
\end{tikzcd}$$

Soit $\alpha \in Spin^c(M)$ qu'on écrit, comme ci-dessus, sous la forme : $\alpha = \alpha_1 \cup j_*(\alpha_2)$. Puisque q_1 est isotopique à l'identité, il agit trivialement sur $Spin^c(M)$, alors : $\alpha = (q_1)_*(\alpha) = (q_1|)_*(\alpha_1) \cup j'_*(\alpha_2)$. Nous obtenons ainsi : $\Omega(\alpha) = \alpha_1 \cup_{j|_\partial \circ h} \alpha_2$ et $\Omega'(\alpha) = (q_1|)_*(\alpha_1) \cup_{j'|_\partial \circ h} \alpha_2$, si bien que : $q_*(\Omega(\alpha)) = \Omega'(\alpha)$.

Nous envisageons maintenant le cas général où G est un clover de degré $d \geq 1$. Remarquons que si G_1 et G_2 sont deux Y-graphes disjoints de M, il découle alors de la définition que, *via* l'identification canonique entre $(M_{G_1})_{G_2}$ et $(M_{G_2})_{G_1}$, les $Spin^c$-structures $(\alpha_{G_1})_{G_2}$ et $(\alpha_{G_2})_{G_1}$ se correspondent.
Soit $Y(G) = \{G_1, \ldots, G_d\}$ la famille associée au clover G de d Y-graphes, numérotés de façon quelconque. Nous posons alors :

$$\Omega_G = \Omega_{G_d} \circ \cdots \circ \Omega_{G_2} \circ \Omega_{G_1}.$$

□

3.2.2. *Y_k-chirurgies spinorielles.* Soient une 3-variété M et un clover G dans M. Le lemme suivant se démontre comme le Lemme 3.5.

LEMME 3.7. *La chirurgie le long de G induit une bijection canonique :*

$$Spin(M) \xrightarrow[\simeq]{\Theta_G} Spin(M_G).$$

Cette correspondance est notée : $\alpha \mapsto \alpha_G$.
En outre, l'application Θ_G est affine au-dessus de $P\left(\Phi_G^{(2)}\right)^{-1} P^{-1}$, où Φ_G est l'isomorphisme d'homologie qui est défini sans équivoque par le diagramme commutatif suivant :

$$\begin{tikzcd}
 & H_1(M) \arrow[dd, "\simeq"', "\Phi_G"] \\
H_1(M \setminus N(G)) \arrow[ur, "k_*"] \arrow[dr, "k'_*"'] & \\
 & H_1(M_G).
\end{tikzcd}$$

Ici, $N(G)$ désigne un voisinage régulier de G dans M, $M \setminus N(G) \xhookrightarrow{k} M$ et $M \setminus N(G) \xhookrightarrow{k'} M_G$ sont les inclusions.

Enfin, dans le cas où $\partial M \neq \varnothing$ *et si* $\sigma \in Spin(\partial M)$ *est telle que* $w_2(M,\sigma) = 0 \in H^2(M, \partial M; \mathbf{Z}_2)$*, nous avons* $\Theta_G\left(Spin(M,\sigma)\right) = Spin(M_G, \sigma)$.

DEFINITION 3.8. La *Spin*-variété (M_G, α_G) est dite obtenue de (M, α) par *Y_k-chirurgie spinorielle*, ou *Y_k^s-chirurgie*, le long de G.
On appelle *Y_k-équivalence spinorielle*, ou *Y_k^s-équivalence*, la relation d'équivalence au sein des *Spin*-variétés de dimension 3, engendrée par les Y_k^s-chirurgies et les *Spin*-difféomorphismes.

REMARQUE 3.9. D'après la Prop. 3.3 b), les bijections Θ_G et Ω_G sont compatibles entre-elles *via* les applications canoniques β.

3.2.3. *Extension du calcul de clovers.* On vérifie dans ce paragraphe que le calcul de clovers s'étend aux cadres *Spin* et *Spin*^{c}.

LEMME 3.10. *Soient* M *une* 3*-variété et* $\alpha \in Spin(M)$*. Soient aussi* G *un clover dans* M *et* $H \subset M$ *un corps en anses plongé contenant* G*. Alors,* via *l'identification canonique :*

$$M_G \cong (M \setminus int(H)) \ \cup \ H_G,$$

$\alpha_G \in Spin(M_G)$ *est l'unique Spin-structure étendant* $\alpha|_{M \setminus int(H)}$.

DÉMONSTRATION. Le fait que α_G étende effectivement la restriction de α à $M \setminus int(H)$ est évident de par la définition même de la Y_k-chirurgie spinorielle le long de G. L'unicité découle de l'injectivité de l'application de restriction :

$$Spin(M) \longrightarrow Spin\left(M \setminus int(H)\right),$$

laquelle est elle-même une conséquence de l'injectivité de :

$$H^1(M; \mathbf{Z}_2) \xrightarrow{k^*} H^1(M \setminus int(H); \mathbf{Z}_2),$$

k désignant l'inclusion. Notons pour cette dernière que : $H^1\left(M, M \setminus int(H); \mathbf{Z}_2\right) \simeq H^1(H, \partial H; \mathbf{Z}_2) \simeq H_2(H; \mathbf{Z}_2) = 0$. □

REMARQUE 3.11. L'analogue du Lemme 3.10 dans le cadre *Spin*^{c} ne vaut que si le clover G est nul-homologue dans M.

PROPOSITION 3.12. *Soient* M *une* 3*-variété,* $\alpha \in Spin(M)$ *ainsi que* G_1 *et* G_2 *deux clovers équivalents de* M.
Alors, les structures $\alpha_{G_1} \in Spin\left(M_{G_1}\right)$ *et* $\alpha_{G_2} \in Spin\left(M_{G_2}\right)$ *se correspondent* via *le difféomorphisme* $M_{G_1} \cong M_{G_2}$ *induit par une équivalence de clovers entre* G_1 *et* G_2.

DÉMONSTRATION. Considérons une équivalence entre les clovers G_1 et G_2 : d'après la Définition 1.10, c'est la donnée d'un corps en anses H contenant à la fois G_1 et G_2, et d'un difféomorphisme $H_{G_1} \xrightarrow[\cong]{f} H_{G_2}$ dont la restriction aux bords est compatible avec les identifications canoniques $\partial H_{G_i} \cong \partial H$; soit aussi $M_{G_1} \xrightarrow{\tilde{f}} M_{G_2}$ le difféomorphisme induit par cette équivalence de clovers. D'après le Lemme 3.10, la structure α_{G_i} est l'unique *Spin*-structure de M_{G_i} étendant $\alpha|_{M \setminus int(H)}$ (pour $i = 1$ et 2). Puisque la restriction de $\tilde{f}$ à $M \setminus int(H)$ est l'identité, on en déduit que $\tilde{f}^*\left(\alpha_{G_2}\right) = \alpha_{G_1}$. □

PROPOSITION 3.13. *Soit* M *une* 3*-variété ainsi que, dans le cas où* $\partial M \neq \varnothing$*, une Spin-structure* σ *sur* ∂M*. Soient aussi* G_1 *et* G_2 *deux clovers équivalents de* M *et* $\alpha \in Spin^c(M)$ *ou* $\alpha \in Spin^c(M, \sigma)$ *suivant que* M *est fermée ou à bord.*
Alors, les structures α_{G_1} *et* α_{G_2} *se correspondent* via *le difféomorphisme* $M_{G_1} \cong M_{G_2}$ *induit par une équivalence de clovers entre* G_1 *et* G_2.

DÉMONSTRATION. Donnons la preuve dans le cas à bord. Soit H un corps en anses contenant à la fois G_1 et G_2, et $H_{G_1} \underset{\cong}{\xrightarrow{f}} H_{G_2}$ une équivalence entre les clovers G_1 et G_2 induisant le difféomorphisme $M_{G_1} \xrightarrow{\tilde{f}} M_{G_2}$ (voir Définition 1.10). On désire montrer que :

$$\tilde{f}^* \left(\alpha_{G_2}\right) = \alpha_{G_1} \in Spin^c \left(M_{G_1}, \sigma\right). \tag{3.9}$$

Prenons aussi $\theta \in Spin(\partial H)$ telle que $w_2(H, \theta) = 0$. L'application de recollement donnée par le Corollaire 2.36 :

$$Spin^c \left(M \setminus int(H), \sigma \dot{\cup} (-\theta)\right) \times Spin^c(H, \theta) \xrightarrow{\cup} Spin^c(M, \sigma)$$

est surjective vu que ∂H est connexe. Il existe donc $\gamma \in Spin^c \left(M \setminus int(H), \sigma \dot{\cup} (-\theta)\right)$ et $\gamma' \in Spin^c(H, \theta)$ telles que $\alpha = \gamma \cup \gamma'$. Vu que :

$$M_{G_i} \cong \left(M \setminus int(H)\right) \cup H_{G_i} \tag{3.10}$$

on a aussi une application de recollement :

$$Spin^c \left(M \setminus int(H), \sigma \dot{\cup} (-\theta)\right) \times Spin^c(H_{G_i}, \theta) \xrightarrow{\cup} Spin^c(M_{G_i}, \sigma),$$

et il découle de la définition de α_{G_i} que : $\alpha_{G_i} = \gamma \cup \gamma'_{G_i}$. Alors $\tilde{f}^* \left(\alpha_{G_2}\right) = \gamma \cup f^* \left(\gamma'_{G_2}\right)$, et l'égalité (3.9) découlera donc de :$f^* \left(\gamma'_{G_2}\right) = \gamma'_{G_1} \in Spin^c \left(H_{G_1}, \theta\right)$.

On peut donc supposer sans restriction de généralité que :

$$w_2(M, \sigma) = 0 \in H^2(M, \partial M; \mathbf{Z}_2).$$

Sous cette hypothèse, on sait grâce à la Proposition 3.12 que (3.9) est satisfaite lorsque α est dans l'image de :

$$Spin \left(M, \sigma\right) \xrightarrow{\beta} Spin^c \left(M, \sigma\right).$$

Soit $\alpha_0 \in Im(\beta)$ et $y \in H^2(M, \partial M)$ tel que $\alpha = y \cdot \alpha_0$. La classe $P^{-1}(y) \in H_1(M)$ est l'image d'une classe $z \in H_1 \left(M \setminus int(H)\right)$ par k_*, $M \setminus int(H) \xhookrightarrow{k} M$ désignant l'inclusion. On note aussi $M \setminus int(H) \xhookrightarrow{k_i} M_{G_i}$ l'inclusion lue dans la décomposition (3.10). Alors, $\alpha_{G_2} = \left(y \cdot \alpha_0\right)_{G_2} = P\Phi_{G_2} P^{-1}(y) \cdot (\alpha_0)_{G_2}$, puis : $\alpha_{G_2} = Pk_{2,*}(z) \cdot (\alpha_0)_{G_2}$. D'où : $\tilde{f}^* \left(\alpha_{G_2}\right) = \tilde{f}^* Pk_{2,*}(z) \cdot (\alpha_0)_{G_1} = P\tilde{f}_*^{-1} k_{2,*}(z) \cdot (\alpha_0)_{G_1}$ $= Pk_{1,*}(z) \cdot (\alpha_0)_{G_1} = P\Phi_{G_1} P^{-1}(y) \cdot (\alpha_0)_{G_1}$. Nous en déduisons finalement que : $\tilde{f}^* \left(\alpha_{G_2}\right) = \left(y \cdot \alpha_0\right)_{G_1} = \alpha_{G_1}$. □

Par conséquent, les résultats *généraux* de la théorie de Goussarov-Habiro qui se démontrent par calcul de clovers, ont des énoncés analogues dans les contextes *Spin* et $Spin^c$.

EXEMPLE 3.14. De la Figure 1.5, nous déduisons par exemple que la relation de Y^s_{k+1}-équivalence (resp. Y^c_{k+1}-équivalence) est plus fine que la relation de Y^s_k-équivalence (resp. Y^c_k-équivalence).

3.3. Invariants de type fini spinoriels au sens de Goussarov-Habiro. Soit $\mathcal{M}^s_0$ une classe de Y^s-équivalence de *Spin*-variétés, éventuellement à bord et soit :

$$\mathcal{F}\left(\mathcal{M}^s_0\right) = \mathbf{Z} \cdot \mathcal{M}^s_0$$

le groupe abélien librement engendré sur $\mathcal{M}^s_0$. Alors, comme en §1.2, on définit successivement le crochet :

$$[(M, \alpha), \Gamma]$$

où $(M, \alpha) \in \mathcal{M}^s_0$ et où Γ est une famille de clovers deux-à-deux disjoints de M, puis les sous-groupes :

$$\mathcal{F}^l_k\left(\mathcal{M}^s_0\right), \quad \mathcal{F}_k\left(\mathcal{M}^s_0\right) \quad \text{et :} \quad \mathcal{F}^l\left(\mathcal{M}^s_0\right)$$

de $\mathcal{F}(\mathcal{M}_0^s)$. Le calcul de clovers valant aussi dans le cas *Spin* (par la Proposition 3.12), les inclusions :

$$\mathcal{F}_{k'}^{l}(\mathcal{M}_0^s) \subset \mathcal{F}_k^{l}(\mathcal{M}_0^s) \subset \mathcal{F}_k^{l'}(\mathcal{M}_0^s)$$

sont satisfaites pour des entiers $1 \leq l \leq l' \leq k \leq k'$. On en déduit que la famille $(\mathcal{F}_k(\mathcal{M}_0^s))_{k\geq 1}$ est une filtration descendante de $\mathcal{F}(\mathcal{M}_0^s)$.

Definition 3.15. Soit A un groupe abélien. Une application $\mathcal{M}_0^s \xrightarrow{f} A$ est un *invariant de type fini spinoriel de degré au plus* k, si son extension $\mathcal{F}(\mathcal{M}_0^s) \xrightarrow{f} A$ s'annule sur $\mathcal{F}_{k+1}(\mathcal{M}_0^s)$. f est un *invariant de type fini de degré* k, si son extension s'annule sur $\mathcal{F}_{k+1}(\mathcal{M}_0^s)$ mais prend une valeur non-nulle sur $\mathcal{F}_k(\mathcal{M}_0^s)$.

En fixant $\mathcal{M}_0^c$ une classe de Y^c-équivalence de $Spin^c$-variétés, qui peuvent être éventuellement à bord (auquel cas les $Spin^c$-structures prises en compte sont relatives à des structures spinorielles), on définit de façon analogue les *invariants de type fini spinoriels complexes*.

Remarquons qu'une classe de Y^s-équivalence $\mathcal{M}_0^s$ détermine de la façon évidente une classe de Y^c-équivalence $\mathcal{M}_0^c$ et une classe de Y-équivalence $\mathcal{M}_0$. On a un diagramme commutatif :

$$\begin{tikzcd} \mathcal{F}(\mathcal{M}_0^s) \arrow[r, "\beta"] \arrow[dr, "\text{oubli}"'] & \mathcal{F}(\mathcal{M}_0^c) \arrow[d, "\text{oubli}"] \\ & \mathcal{F}(\mathcal{M}_0). \end{tikzcd}$$

Tous ces homomorphismes respectent les filtrations $\mathcal{F}_k(-)$. Ainsi, un invariant de type fini de degré k détermine un invariant de type fini spinoriel complexe de degré k, et un invariant de type fini spinoriel complexe de degré k détermine un invariant de type fini spinoriel de degré k.

Enfin, comme expliqué au §1.2 pour la théorie brute, l'étude de la théorie de Goussarov-Habiro raffinée *Spin* ou $Spin^c$ scinde en deux parties.

Remarque 3.16. Le fait que leur théorie puisse être raffinée aux variétés munies de *Spin*-structures était déjà mentionné par Goussarov dans [**Go**] et Habiro dans [**Hr**].

Remarque 3.17. Nos techniques (en particulier, les Propositions 3.3 et 3.2), permettent de raffiner aux cadres *Spin* et $Spin^c$ les théories d'invariants de type fini associées à des mouvements élémentaires autres que la Y-chirurgie, tant que ceux-ci se définissent par des twists le long de surfaces plongées par des éléments de leur groupe de Torelli. Voici des exemples de tels mouvements :

(1) la chirurgie sur les "blinks" (voir [**GGP**, §1.7]).

(2) la chirurgie sur les "boundary links" (voir [**GGP**, §1.6]).

(3) le mouvement de Whitehead (voir [**Mt**] et [**MP**, §4.6]).

CHAPITRE 2

Formes quadratiques et invariants de type fini des *Spin*-variétés fermées de dimension 3

1. Introduction

The notion of Borromean surgery was introduced by Matveev in [**Mt**] as an example of what he called a $\mathcal{V}$-surgery. Since then, this transformation has become the elementary move of Goussarov-Habiro finite type invariants theory for oriented 3-manifolds ([**Hr**], [**Go**], [**GGP**]). Matveev showed that the equivalence relation, among closed oriented 3-manifolds, generated by Borromean surgery is characterized by the pair:

$$(\beta_1(M),\ \text{isomorphism class of } \lambda_M),$$

where $\beta_1(M)$ is the first Betti number of a 3-manifold M and

$$TH_1(M;\mathbf{Z}) \otimes TH_1(M;\mathbf{Z}) \xrightarrow{\lambda_M} \mathbf{Q}/\mathbf{Z}$$

is its torsion linking form, where $TH_1(M)$ denotes the torsion subgroup of $H_1(M)$. This result gives a characterization of degree 0 invariants in Goussarov-Habiro theory for closed oriented 3-manifolds.

As mentioned by Habiro and Goussarov, their finite type invariants theory (in short: "FTI theory") makes sense also for 3-manifolds with spin structure because Borromean surgeries work well with spin structures (see §3). So, the question is: *what is the "spin" analogue of Matveev's theorem?*
For each closed spin 3-manifold (M,σ), a quadratic form

$$TH_1(M;\mathbf{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbf{Q}/\mathbf{Z}$$

can be defined by many ways (see [**LL**], [**MS**], and also [**T2**], [**Gi**]). The bilinear form associated to $\phi_{M,\sigma}$ is λ_M. Its Gauss-Brown invariant is equal to $-R_M(\sigma)$ modulo 8, where

$$Spin(M) \xrightarrow{R_M} \mathbf{Z}_{16}$$

is the Rochlin function of M, sending a spin structure σ of M to the modulo 16 signature of a spin 4-manifold which spin-bounds (M,σ). The main result is the following refinement of Matveev's theorem:

Theorem 1.1. *Let (M,σ) and (M',σ') be connected closed spin 3-manifolds. Then, the following assertions are equivalent:*

1. *(M,σ) and (M',σ') can be obtained from one another by spin Borromean surgeries,*
2. *there exists a homology isomorphism $f : H_1(M;\mathbf{Z}) \longrightarrow H_1(M';\mathbf{Z})$ such that:*
$$\phi_{M,\sigma} = \phi_{M',\sigma'} \circ f|,$$
3. *$R_M(\sigma) = R_{M'}(\sigma')$ modulo 8 and there exists a homology isomorphism $f : H_1(M;\mathbf{Z}) \longrightarrow H_1(M';\mathbf{Z})$ such that:*
$$\lambda_M = \lambda_{M'} \circ (f|)^{\otimes 2}.$$

The equivalence between assertions 2 and 3 will be the topological statement of an algebraic fact: a nondegenerate quadratic form on a finite Abelian group is determined, up to isomorphism, by its associated bilinear form and its Gauss-Brown invariant.

In §2 we recall Matveev's notion of $\mathcal{V}$-surgery. With this background, we then recall the definition of Borromean surgery, and give equivalent descriptions of other authors.
In §3, we clarify how all of these notions have to be understood in the spin case: in particular, spin Borromean surgeries are introduced. As a motivation to Theorem 1.1, FTI for spin 3-manifolds, in the sense of Habiro and Goussarov, are then defined: the Rochlin invariant is shown to be a finite type degree 1 invariant. It should be mentioned that Cochran and Melvin have proposed a different FTI theory in [**CM**], and have also refined their theory to the case of spin manifolds.
§4 is of an algebraic nature. We recall some definitions and results about quadratic forms on finite Abelian groups. We also prove the above mentioned algebraic fact: the proof makes use of Kawauchi-Kojima classification of linking pairings.
§5 is the topological cousin of the former: we review the quadratic form $\phi_{M,\sigma}$. Starting from Turaev 4-dimensional definition in [**T2**], we then give an intrinsic definition for $\phi_{M,\sigma}$ (no reference to dimension 4).
§6 is devoted to the proof of Theorem 1.1. It goes as a refinement of the original proof by Matveev for the "unspun" case. Last section will give some of its applications.

2. Borromean surgeries and equivalent moves

First of all, we want to recall the *unifying* idea of $\mathcal{V}$-surgery by Matveev in [**Mt**]. This will allow us to have a more conceptual view of Borromean surgeries.

2.1. Review of Matveev $\mathcal{V}$-surgeries. We begin with some general definitions.

DEFINITION 2.1. A *Matveev triple* is a triple of oriented 3-manifolds:

$$\mathcal{V} = (V, V_1, V_2), \tag{2.1}$$

where V is closed and is the union of $-V_1$ and V_2 along their common boundary $\partial V_1 = \partial V_2$, as depicted in Figure 2.1.
The triple $(-V, V_2, V_1)$ is called the *inverse* of $\mathcal{V}$ and is denoted by $\mathcal{V}^{-1}$.

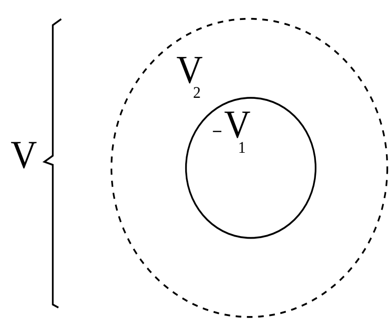

Figure 2.1: What will be removed and what will be glued during the surgery.

Let now M be a closed oriented 3-manifold and let $j : V_1 \longrightarrow M$ be an orientation-preserving embedding. Form the following closed oriented 3-manifold:

$$M' = M \setminus int\,(j(V_1)) \cup_{j|_\partial} V_2. \tag{2.2}$$

DEFINITION 2.2. With the above notations, M' is said to be obtained from M by *$\mathcal{V}$-surgery along j*.

Note that if k denotes the embedding of V_2 in M', then M is obtained from M' by $\mathcal{V}^{-1}$-surgery along k.

DEFINITION 2.3. Two Matveev triples $\mathcal{V}$ and $\mathcal{V}'$ are said to be *equivalent* if there exists an orientation-preserving diffeomorphism from V to V' sending $-V_1$ to $-V_1'$, and V_2 to V_2'.

Note that, if the triples $\mathcal{V}$ and $\mathcal{V}'$ are equivalent, then they have the same surgery effect.

EXAMPLE 2.4. Let Σ_g denote the genus g closed oriented surface and let H_g be the genus g oriented handlebody. Then, each orientation-preserving diffeomorphism $f : \Sigma_g \longrightarrow \Sigma_g$ leads to a triple:

$$\mathcal{V}_f := \left((-H_g) \cup_f H_g, H_g, H_g\right).$$

A $\mathcal{V}_f$-surgery amounts to "twist" an embedded genus g handlebody by f. For instance, from the standard genus one Heegaard decomposition of $\mathbf{S}^3$, integral Dehn surgery is recovered.

2.2. Review of Borromean surgeries. The original Matveev's point of view was:

DEFINITION 2.5. A *Borromean surgery* is a $\mathcal{B}$-surgery with:

$$\mathcal{B} = \left(B := (-B_1) \cup B_2, B_1, B_2\right),$$

where the "halves" B_1 and B_2 are obtained from the genus 3 handlebody by surgery on three-component framed links as shown in Figure[1] 2.2.

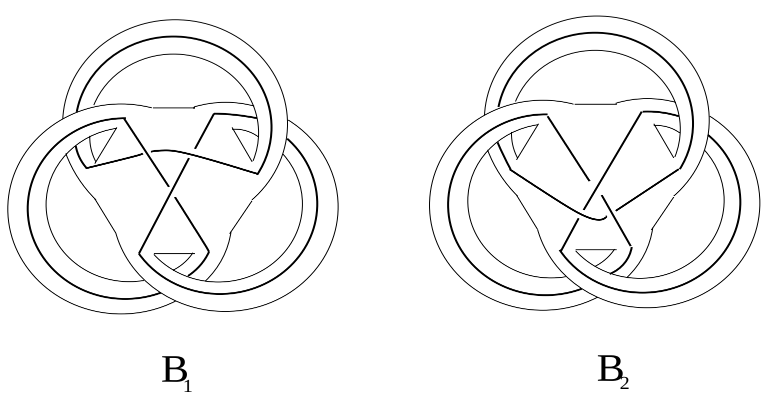

Figure 2.2: The triple $\mathcal{B}$.

We now recall Goussarov's notion of *Y-surgery* in [**Go**]. This move is equivalent to the *A_1-move* of Habiro in [**Hr**].

[1] Blackboard framing convention is used.

DEFINITION 2.6. A *Y-graph* G in a closed oriented 3-manifold M is an (unoriented) embedding of the surface drawn in Figure 2.3, together with its decomposition between *leaves*, *edges* and *node*.
The closed oriented 3-manifold *obtained from M by Y-surgery along G* is:

$$M_G := (M \setminus int(N(G))) \cup (H_3)_L,$$

where $N(G) \cong_+ H_3$ is a regular neighbourhood of G in M, and $(H_3)_L$ is the surgered handlebody on the six-component link L drawn on Figure 2.4.
We call *Y-equivalence* the equivalence relation among closed oriented 3-manifolds generated by orientation-preserving diffeomorphisms and Y-surgeries.

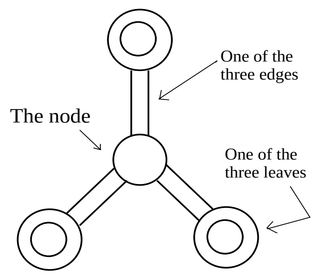

Figure 2.3: A Y-graph.

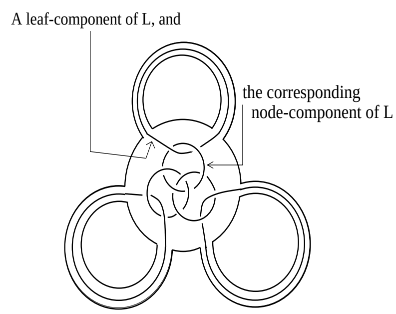

Figure 2.4: The surgery meaning of a Y-graph.

Note that a Y-surgery is a $\mathcal{Y}$-surgery if we call $\mathcal{Y}$ the triple:

$$\mathcal{Y} = (Y := (-H_3) \cup (H_3)_L, H_3, (H_3)_L)\,,$$

the corresponding Y-graph gives the place where the $\mathcal{Y}$-surgery must be performed.

LEMMA 2.7. *The Matveev triples $\mathcal{B}$ and $\mathcal{Y}$ are equivalent. Thus, Borromean surgery is equivalent to Y-surgery.*

PROOF. We will show that both of the triples $\mathcal{B}$ and $\mathcal{Y}$ are equivalent to a triple $\mathcal{V}_h$, defined by an orientation-preserving diffeomorphism $h : \Sigma_3 \longrightarrow \Sigma_3$.
We start with the "half" $(H_3)_L$ of Figure 2.4: handle-sliding of each node-component over the corresponding leaf-component, followed by some isotopies of framed links gives Figure 2.5, where only part of the link is drawn. Up to a $(+1)$-framing correction, the three depicted components a_i can be normally pushed off at once towards the boundary: we obtain three disjoint curves α_i on ∂H_3. Note that during this push-off, none of the three components b_i is intersected. Then, the components b_i

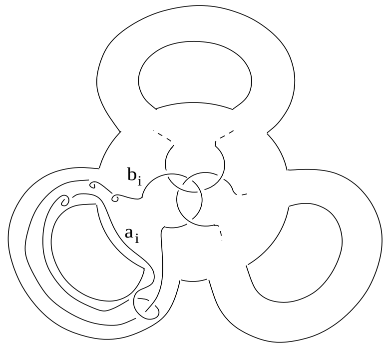

Figure 2.5: The link L of Figure 2.4 after some Kirby's moves.

can also be pushed off so that the framing correction is now -1: the result is a family of three disjoint curves β_i. After a convenient isotopy of the handles, the curves α_i and β_i can be depicted as on Figure 2.6.

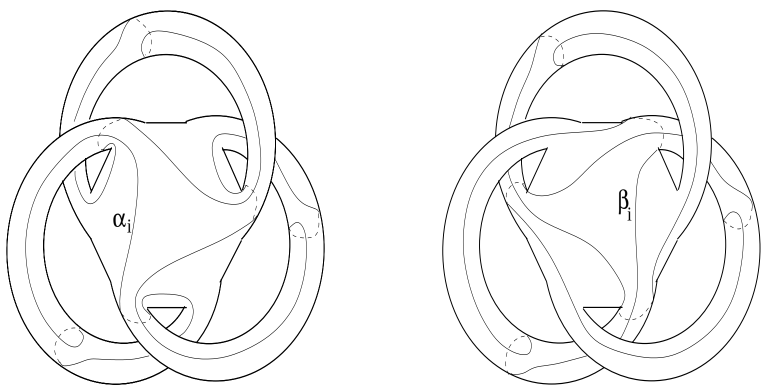

Figure 2.6: The curves α_i (on the lefthand side) and β_i (on the righthand side) resulting from the push-off of the corresponding knots a_i and b_i of Figure 2.5.

We define $h := h_b^{-1} \circ h_a$, where h_a and h_b are the following composites of (commuting) Dehn twists:

$$h_a = \prod_{i=1}^{3} \tau_{\alpha_i} \quad \text{and} \quad h_b = \prod_{i=1}^{3} \tau_{\beta_i}.$$

According to the Lickorish trick [**Li**, proof of Theorem 2], a $\mathcal{Y}$-surgery is therefore equivalent to a $\mathcal{V}_h$-surgery.

On the other hand, from Figure 2.2, we deduce that a $\mathcal{B}$-surgery is equivalent to a $\mathcal{V}_{h'}$-surgery where $h' := h_b \circ h_a^{-1}$. Let m_i denote the meridian of the ith handle of H_3, for $i = 1, 2, 3$. Then, the equation:

$$k^{-1} = k \circ \prod_{i=1}^{3} \tau_{m_i}^2 \tag{2.3}$$

holds for both h_a and h_b so that $h = h'$. □

REMARK 2.8 (Fundamental remark). Note that, in the proof of Lemma 2.7, the curve α_i is homologous in the surface Σ_3 to the corresponding curve β_i (look at Figure 2.6). As a consequence, the diffeomorphism $h : \Sigma_3 \longrightarrow \Sigma_3$ belongs to the Torelli group. In the sequel, we will call h the *Borromean diffeomorphism.*

In order to have a complete understanding of these equivalent triples $\mathcal{B}$, $\mathcal{Y}$ or $\mathcal{V}_h$, it remains to recognize their underlying closed 3-manifolds.

LEMMA 2.9. *The closed 3-manifolds B and Y, respectively defined by the triples $\mathcal{B}$ and $\mathcal{Y}$, are both homeomorphic to the 3-torus $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$.*

PROOF. According to Lemma 2.7, B and Y are diffeomorphic. Let us identify Y. Recall that Y was defined as:

$$Y = (-H_3) \cup (H_3)_L. \tag{2.4}$$

Write L as $L_a \dot{\cup} L_b$ where L_a (resp. L_b) is the sublink containing the leaf(resp. node)-components. Note that L_b can be isotoped in $Y \setminus L_a$ to some Borromean rings contained in a 3-ball disjoint from L_a: make L_b leave the handlebody where it was lying, towards the handlebody with the minus sign in (2.4). So Y is obtained from $(-H_3) \cup (H_3)_{L_a}$ by surgery on some Borromean rings contained in a little 3-ball. The lemma then follows from the fact that the latter is nothing but $\mathbf{S}^3$. □

3. Spin Borromean surgeries

We now go into the world of spin 3-manifolds. We refer to [**Ki**, Chapter IV] for an introduction to spin structures. As a warm up, we recall a few facts in the next subsection.

3.1. Gluing of spin structures. Let $n \geq 2$, and let M be a compact smooth oriented n-manifold endowed with a Riemannian metric. Its bundle of oriented orthonormal frames will be denoted by FM: it is a principal $SO(n)$-bundle with total space $E(FM)$ and with projection p.
Recall that if M is spinnable, $Spin(M)$ can be thought of as:

$$Spin(M) = \left\{\sigma \in H^1\left(E(FM);\mathbf{Z}_2\right) \ : \ \sigma|_{\text{fiber}} \neq 0 \in H^1\left(SO(n);\mathbf{Z}_2\right)\right\},$$

and is essentially independant of the metric. The set $Spin(M)$ is then an affine space over $H^1(M;\mathbf{Z}_2)$, the action being defined by:

$$\forall x \in H^1(M;\mathbf{Z}_2),\ \forall \sigma \in Spin(M),\ x \cdot \sigma := \sigma + p^*(x).$$

LEMMA 3.1. *For $i = 1, 2$, let M_i be a compact smooth oriented n-manifold and let S_i be a submanifold of ∂M_i with orientation induced by M_i. Let also $f : S_2 \longrightarrow S_1$ be an orientation-reversing diffeomorphism and let $M = M_1 \cup_f M_2$. Assume that M_1 and M_2 are spinnable, that S_2 is connected, and that the set:*

$$J = \left\{(\sigma_1, \sigma_2) \in Spin(M_1) \times Spin(M_2) \ : \ f^*\left(-\sigma_1|_{S_1}\right) = \sigma_2|_{S_2}\right\}$$

is not empty. Then, M is spinnable and the restriction map:

$$\left\{ \begin{array}{ccc} Spin(M) & \xrightarrow{r} & Spin(M_1) \times Spin(M_2) \\ \sigma & \longmapsto & \left(\sigma|_{M_1}, \sigma|_{M_2}\right), \end{array} \right.$$

is injective with J as image.

PROOF. For $i = 1$ or 2, let F^+S_i be the principal $SO(n)$-bundle derived from FS_i and the inclusion of groups:

$$SO(n-1) \overset{g}{\hookrightarrow} SO(n).$$

We still denote by g the canonical map from FS_i to F^+S_i. Then,

$$H^1\left(E\left(F^+S_i\right);\mathbf{Z}_2\right) \xrightarrow{g^*} H^1\left(E\left(FS_i\right);\mathbf{Z}_2\right)$$

is an isomorphism. The bundle F^+S_i can be identified with $FM_i|_{S_i}$. In particular, there is an inclusion map $k_i : E\left(F^+S_i\right) \hookrightarrow E(FM_i)$.

Moreover, the diffeomorphism f induces a further identification:

$$E\left(F^+S_2\right) \underset{\simeq}{\xrightarrow{f}} E\left(F^+S_1\right),$$

such that the total space $E(FM)$ is homeomorphic to the gluing:

$$E(FM_1) \cup_{k_1 f k_2^{-1}} E(FM_2).$$

We denote by j_i the corresponding inclusion of $E(FM_i)$ into $E(FM)$. But now, by the Mayer-Vietoris sequence, we have:

$$\begin{array}{ccc} H^1\left(E(FM);\mathbf{Z}_2\right) & \xrightarrow{(j_1^*,j_2^*)} & H^1\left(E(FM_1);\mathbf{Z}_2\right) \oplus H^1\left(E(FM_2);\mathbf{Z}_2\right) \\ & & \Big\downarrow (k_1\circ f)^* - k_2^* \\ & & H^1\left(E\left(F^+S_2\right);\mathbf{Z}_2\right). \end{array}$$

Note that $g^* \circ (k_1 \circ f)^*$ sends each $\sigma_1 \in Spin(M_1)$ to $f^*(-\sigma_1|_{S_1})$, while $g^* \circ k_2^*$ sends each $\sigma_2 \in Spin(M_2)$ to $\sigma_2|_{S_2}$. Note also that, since S_2 is connected, the map (j_1^*, j_2^*) is injective. The whole lemma then follows from these two remarks and from the exactness of the Mayer-Vietoris sequence. □

Definition 3.1. With the notations and hypothesis of Lemma 3.1, for each $(\sigma_1,\sigma_2) \in J$, the unique spin structure of M sent by r to (σ_1,σ_2) is called the *gluing* of the spin structures σ_1 and σ_2, and is denoted by $\sigma_1 \cup \sigma_2$.

3.2. Spin $\mathcal{V}$-surgeries. In some cases, a Matveev $\mathcal{V}$-surgery, whose definition has been recalled in §2.1, makes sense for spin 3-manifolds.

Definition 3.2. A Matveev triple $\mathcal{V} = (V, V_1, V_2)$ is said to be *spin-admissible*, if $\partial V_1 = \partial V_2$ is connected, and if the maps:

$$H^1(V;\mathbf{Z}_2) \longrightarrow H^1(V_1;\mathbf{Z}_2) \quad \text{and} \quad H^1(V;\mathbf{Z}_2) \longrightarrow H^1(V_2;\mathbf{Z}_2),$$

induced by inclusions, are isomorphisms.

Suppose now that $\mathcal{V}$ is a spin-admissible triple. Note that the restriction maps:

$$Spin(V) \xrightarrow{r_1} Spin(-V_1) \text{ and } Spin(V) \xrightarrow{r_2} Spin(V_2)$$

are then bijective. Let also M be a closed oriented 3-manifold and let $j : V_1 \longrightarrow M$ be an orientation-preserving embedding. As in §2.1, we denote by M' the result of the $\mathcal{V}$-surgery along j, and we want to define a *canonical bijection*:

$$Spin(M) \xrightarrow{\Theta_{j,\mathcal{V}}} Spin(M').$$

First, the embedding j allows us to define the following map:

$$\left\{\begin{array}{ccc} Spin(M) & \xrightarrow{\Upsilon_{j,\mathcal{V}}} & Spin(V) \\ \sigma & \longmapsto & r_1^{-1}(-j^*(\sigma)). \end{array}\right.$$

From formula (2.2) and from Definition 3.1, we can define $\Theta_{j,\mathcal{V}}(\sigma)$ as the following gluing:

$$\Theta_{j,\mathcal{V}}(\sigma) := \sigma|_{M\setminus int(j(V_1))} \cup r_2\left(\Upsilon_{j,\mathcal{V}}(\sigma)\right).$$

The inverse of $\Theta_{j,\mathcal{V}}$ is $\Theta_{k,\mathcal{V}^{-1}}$, where k denotes the embedding of V_2 in M' .

DEFINITION 3.3. With the above notations, the spin manifold $(M', \Theta_{j,\mathcal{V}}(\sigma))$ is said to be obtained from (M, σ) by *spin $\mathcal{V}$-surgery along j*.

EXAMPLE 3.2. Let $f : \Sigma_g \longrightarrow \Sigma_g$ be an orientation-preserving diffeomorphism. Denote by K the lagrangian subspace of $H_1(\Sigma_g; \mathbf{Z}_2)$ spanned by the meridians. Then, as can be easily verified, the triple $\mathcal{V}_f$ of Example 2.4 is spin-admissible if and only if $f_*(K) = K$. For instance, this condition is satisfied when f belongs to the Torelli modulo 2 group.

LEMMA 3.3. *In the particular case of Example 3.2, that is when $\mathcal{V} = \mathcal{V}_f$ with $f_*(K) = K$, then for each $\sigma \in Spin(M)$, $\Theta_{j,\mathcal{V}}(\sigma)$ is the unique spin structure of M' extending $\sigma|_{M \setminus int(j(V_1))}$.*

PROOF. In that case, V_2 is a handlebody and so $H^1(M', M \setminus int\,(j(V_1)); \mathbf{Z}_2)$ is zero. Therefore, the restriction map $Spin(M') \longrightarrow Spin(M \setminus int(j(V_1)))$ is injective. □

3.3. Definition of Y^s-surgeries. From Lemma 2.7 and from Remark 2.8 above, we have learnt that both of the triples $\mathcal{B}$ and $\mathcal{Y}$ are equivalent to the triple $\mathcal{V}_h$ where h is the Borromean diffeomorphism which belongs to the Torelli group. So, by Example 3.2, they are spin-admissible and the following definition makes sense:

DEFINITION 3.4. A *Y^s-surgery*, or equivalently a *spin Borromean surgery*, is the surgery move among closed spin 3-manifolds defined equivalently by the triples $\mathcal{Y}$ or $\mathcal{B}$. We call *Y^s-equivalence* the equivalence relation among them generated by spin diffeomorphisms and Y^s-surgeries.

Let (M, σ) be a closed spin 3-manifold and let G be a Y-graph in M. The Y^s-surgery along G gives a new spin manifold which will be denoted by:

$$(M_G, \sigma_G).$$

Let $j : H_3 \longrightarrow N(G)$ be an embedding of the genus 3 handlebody onto a regular neighbourhood of G in M. Then,

$$M_G \cong M \setminus int\,(N(G)) \cup_{j|_\partial \circ h} H_3, \tag{3.1}$$

and according to Lemma 3.3, σ_G is the unique spin structure of M_G extending $\sigma|_{M \setminus int(N(G))}$.

3.4. Goussarov-Habiro FTI theory for spin 3-manifolds.

LEMMA 3.5. *Let (M, σ) be a closed spin 3-manifold and let G and H be disjoint Y-graphs in M. Then, up to diffeomorphism of manifolds with spin structure,*

$$((M_G)_H, (\sigma_G)_H) = ((M_H)_G, (\sigma_H)_G).$$

PROOF. The equality $(M_G)_H = (M_H)_G$ is obvious. By construction, both of $(\sigma_G)_H$ and $(\sigma_H)_G$ are extensions of $\sigma|_{M \setminus (N(G) \cup N(H))}$. The lemma then follows from the fact that the restriction map:

$$Spin\,((M_G)_H) \longrightarrow Spin\,(M \setminus (N(G) \cup N(H)))$$

is injective since:

$$H^1\,((M_G)_H, M \setminus (N(G) \cup N(H)); \mathbf{Z}_2) = 0.$$

□

Let $S = \{G_1, \ldots, G_s\}$ be a family of disjoint Y-graphs in a closed 3-manifold M with spin structure σ. Lemma 3.5 says that *Y^s-surgery along the family S* is well-defined. We denote the result by (M_S, σ_S). Following Habiro and Goussarov definition of a finite type invariant ([**Hr**], [**Go**]), we can now define:

DEFINITION 3.4. Let A be an Abelian group and let λ be an A-valued invariant of 3-manifolds with spin structure. Then, λ is an *invariant of degree at most* n if for any closed spin 3-manifold (M,σ) and any family S of at least $n+1$ Y-graphs in M, the following identity holds:

$$\sum_{S'\subset S} \lambda(M_{S'},\sigma_{S'}) = 0 \in A, \tag{3.2}$$

where the sum is taken over all subfamilies S' of S. Moreover, λ is *of degree* n if it is of degree at most n, but is not of degree at most $n-1$.

REMARK 3.6. Note that the degree 0 invariants are precisely those invariants which are constant on each Y^s-equivalence class. So, the refined Matveev theorem will quantify how powerful they can be.

The next subsection will provide us some examples of invariants which are finite type in the sense of Definition 3.4.

3.5. Rochlin invariant under Y^s-surgery.

PROPOSITION 3.7. *Let (M,σ) be a closed spin 3-manifold, and let G be a Y-graph in M. Then, the following formula holds:*

$$R_{M_G}(\sigma_G) = R_M(\sigma) + R_{\mathbf{S}^1\times\mathbf{S}^1\times\mathbf{S}^1}(\Upsilon_G(\sigma)) \in \mathbf{Z}_{16},$$

where the map $\Upsilon_G : Spin(M) \longrightarrow Spin(\mathbf{S}^1\times\mathbf{S}^1\times\mathbf{S}^1)$, induced by the $\mathcal{Y}$-surgery along G, has been defined in §3.2.

PROOF. According to Lemma 2.9, we can think of the 3-torus as:

$$\mathbf{S}^1\times\mathbf{S}^1\times\mathbf{S}^1 = (-H_3)\cup_h H_3. \tag{3.3}$$

The surgered manifold M_G will be thought of concretely as in (3.1).
Pick a spin 4-manifold W spin-bounded by (M,σ), and a spin 4-manifold H spin-bounded by the 3-torus with $\Upsilon_G(\sigma)$ as a spin structure. Glue the "generalized" handle H to W along the first handlebody of the 3-torus in decomposition (3.3), using j as glue. We obtain a 4-manifold W'. Orient W' coherently with H and W, and then give to W' the spin structure obtained by gluing those of H and W (see Definition 3.1). It follows from definitions that the spin-boundary of W' is (M_G,σ_G).
According to Wall's theorem on non-additivity of the signature (see [**W3**]), we have:

$$sgn(W') = sgn(W) + sgn(H) - \text{ correcting term.} \tag{3.4}$$

The involved correcting term is the signature of a real bilinear symmetric form explicitly described by Wall. It is defined by means of the intersection form in Σ_3, with domain:

$$V = \frac{A\cap(B+C)}{A\cap B + A\cap C},$$

where A,B,C are subspaces of $H_1(\Sigma_3;\mathbf{R})$ defined to be respectively the kernels of:

$$H_1(\Sigma_3;\mathbf{R}) \xrightarrow{(j|_\partial)_*} H_1(\partial N(G);\mathbf{R}) \xrightarrow{i_*} H_1(M\setminus int(N(G));\mathbf{R}),$$

$$H_1(\Sigma_3;\mathbf{R}) \xrightarrow{\qquad\qquad i_*\qquad\qquad} H_1(H_3;\mathbf{R}),$$

$$\text{and:}\quad H_1(\Sigma_3;\mathbf{R}) \xrightarrow{(h^{-1})_*} H_1(\Sigma_3;\mathbf{R}) \xrightarrow{\quad i_*\quad} H_1(H_3;\mathbf{R}).$$

No matter who is A, since the Borromean diffeomorphism h lies in the Torelli group, we certainly have $B=C$. The space V then vanishes and so does the correcting term. The announced equality then follows by taking equation (3.4) modulo 16. □

COROLLARY 3.8. *The Rochlin invariant is a degree* 1 *invariant of closed spin* 3*-manifolds for Goussarov-Habiro theory, and its modulo* 8 *reduction is of degree* 0.

REMARK 3.9. In Cochran-Melvin theory, the Rochlin invariant is a degree 3 finite type invariant (see [**CM**, Prop. 6.2]).

PROOF OF COROLLARY 3.8. Last statement is clear from Proposition 3.7 and from the fact that the Rochlin function of the 3-torus takes values in $\{0,8\} \subset \mathbf{Z}_{16}$. Let us show that the Rochlin invariant is at most of degree 1. Take a closed spin 3-manifold (M,σ) and two disjoint Y-graphs G and H in M. According to Proposition 3.7, in order to show that:

$$R_M(\sigma) - R_{M_G}(\sigma_G) - R_{M_H}(\sigma_H) + R_{M_{G,H}}(\sigma_{G,H}) = 0, \tag{3.5}$$

it suffices to show that:

$$\Upsilon_G(\sigma) = \Upsilon_G(\sigma_H) \in Spin(\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1), \tag{3.6}$$

where the left Υ_G is defined by $G \subset M$ and the right Υ_G is defined by $G \subset M_H$. But this follows from definition of the maps Υ_G and from the fact that σ_H extends $\sigma|_{M\setminus N(H)}$.
It remains to show that the Rochlin invariant is not of degree 0 (and so it will be "exactly" of degree 1). For instance, all of the spin structures of the 3-torus are related one to another by Y^s-surgeries (*Cf* Example 3.14 below) and so are not distinguished one to another by degree 0 invariants. But Rochlin distinguishes one of them from the others. □

3.6. Y^s-surgeries through surgery presentations on $\mathbf{S}^3$.

We first fix some notations. We call V_L the 3-manifold obtained from $\mathbf{S}^3$ by surgery along a ordered oriented framed link $L = (L_1, \dots, L_l)$ of length l, and W_L the corresponding 4-manifold obtained from $\mathbf{B}^4$ by attaching 2-handles and sometimes called the *trace* of the surgery. Let also $B_L = (b_{ij})_{i,j=1,\dots,l}$ be the linking matrix of L.
Recall that $H_2(W_L;\mathbf{Z})$ is free Abelian of rank l. For each $i = 1, \dots, l$, choose a Seifert surface of L_i in $\mathbf{S}^3$ and push it off into the interior of $\mathbf{B}^4$: denote the result by P_i. Then, glue P_i to the core of the ith 2-handle to obtain a closed surface S_i. A basis of $H_2(W_L;\mathbf{Z})$ is then given by $([S_1], \dots, [S_l])$.

We now recall a nice combinatorial way to describe spin structures of V_L. This is by means of the so-called "characteristic solutions of B_L" or, equivalently, "characteristic sublinks of L".
A vector $s \in (\mathbf{Z}_2)^l$ or, equivalently, the sublink of L containing the components L_i such that $s_i = 1$, are said to be *characteristic* if the following equation is satisfied:

$$\forall i = 1, \dots, l\ , \quad \sum_{j=1}^{l} b_{ij} \cdot s_j = b_{ii} \in \mathbf{Z}_2. \tag{3.7}$$

We denote by $\mathcal{S}_L$ the subset of $(\mathbf{Z}_2)^l$ comprising the characteristic solutions of B_L. There is a bijection:

$$Spin(V_L) \xrightarrow{\simeq} \mathcal{S}_L$$

which is defined by the following composition:

$$Spin(V_L) \xrightarrow{o} H^2(W_L, V_L; \mathbf{Z}_2) \underset{\simeq}{\xrightarrow{P}} H_2(W_L; \mathbf{Z}_2) \underset{\simeq}{\longrightarrow} (\mathbf{Z}_2)^l$$

where o sends any $\sigma \in Spin(V_L)$ to the obstruction to extending σ to the whole of W_L, P is the Poincaré duality isomorphism and the last map is defined by the basis $([S_1], \dots, [S_l])$. With this combinatorial description, Kirby's theorem can be

refined to closed spin 3-manifolds (see [**Bl**]).
The following lemma, more general than needed, will allow us to enunciate in those terms the effect of a Y^s-surgery.

LEMMA 3.10. *Let $L \cup K$ be the ordered union of two ordered oriented framed links in $\mathbf{S}^3$ and let $H \subset S^3$ be an embedded handlebody such that K is contained in the interior of H, L is disjoint from H, and H_K is a $\mathbf{Z}_2$-homology handlebody.*

Suppose that $\sigma \in Spin\,(V_{L\cup K})$ is represented by a characteristic solution $s \in (\mathbf{Z}_2)^{l+k}$ of $B_{L\cup K}$ satisfying the following two properties:

(1) *$(s_1, \dots, s_l) \in (\mathbf{Z}_2)^l$ is a characteristic solution of B_L,*
(2) *for all $i \in \{1, \dots, k\}$ such that $s_{l+i} \neq 0$, the component K_i bounds a Seifert surface* within *H.*

Then, the restricted spin structure:

$$\sigma_| \in Spin(V_{L\cup K} \setminus int(H_K)) = Spin(V_L \setminus int(H)),$$

extends to the spin structure of V_L represented by $(s_1, \dots, s_l) \in (\mathbf{Z}_2)^l$.

PROOF. In the following, all (co)homology groups are assumed to be with coefficients in $\mathbf{Z}_2$. We use the above fixed notations.
Let us consider the map:

$$Spin\,(V_L \setminus H) \xrightarrow{o} H^2\,(W_L, V_L \setminus H)\,,$$

where $o(\alpha)$ is the obstruction to extending any $\alpha \in Spin(V_L \setminus H)$ to the whole of W_L. Let also $\delta^* : H^1\,(V_L \setminus H) \longrightarrow H^2\,(W_L, V_L \setminus H)$ denote the connecting homomorphism for the pair $(W_L, V_L \setminus H)$. Note that the following equation holds:

$$\forall x \in H^1\,(V_L \setminus H)\,, \forall \alpha \in Spin(V_L \setminus H),\ o(x \cdot \alpha) = o(\alpha) + \delta^*(x). \tag{3.8}$$

Since δ^* is injective, it follows that o is injective.
The same map o can be defined for V_L relatively to W_L, and for $V_{L\cup K}$ and $V_{L\cup K} \setminus H_K$ relatively to $W_{L\cup K}$. We have thus the following commutative diagram:

$$\begin{tikzcd}
Spin(V_{L\cup K}) \arrow[r, hook, "o"] \arrow[d, hook] & H^2(W_{L\cup K}, V_{L\cup K}) \arrow[r, "\simeq", "P"'] \arrow[d, hook] & H_2(W_{L\cup K}) \arrow[d, hook] \arrow[dd, dashed, bend left=60, "r"] \\
Spin(V_{L\cup K} \setminus H_K) \arrow[r, hook, "o"] \arrow[d, equal] & H^2(W_{L\cup K}, V_{L\cup K} \setminus H_K) \arrow[r, "\simeq", "P"'] \arrow[d] & H_2(W_{L\cup K}, H_K) \\
Spin(V_L \setminus H) \arrow[r, hook, "o"] & H^2(W_L, V_L \setminus H) \arrow[r, "\simeq", "P"'] & H_2(W_L, H) \\
Spin(V_L) \arrow[r, hook, "o"] \arrow[u, hook] & H^2(W_L, V_L) \arrow[r, "\simeq", "P"'] \arrow[u, hook] & H_2(W_L) \arrow[u, hook, "i_*"']
\end{tikzcd}$$

where the letter P stands for a Poincaré duality isomomorphism, the vertical arrows are induced by inclusions and the map r is defined by planar commutativity.
From intersection theory, we deduce that:

$$\begin{cases} \forall i \in \{1, \dots, l\}, & r\,([S_i]) = [S_i], \\ \forall i \in \{l+1, \dots, l+k\}, & r\,([S_i]) = [P_i]. \end{cases} \tag{3.9}$$

Let now σ be a spin structure of $V_{L\cup K}$ such that the corresponding characteristic solution s of $B_{L\cup K}$ satisfies the conditions 1 and 2 of the lemma.
We define $\tilde{s} := (s_1, \dots, s_l) \in (\mathbf{Z}_2)^l$. By hypothesis 1, there exists a unique spin structure $\tilde{\sigma}$ of V_L with $\tilde{s}$ as associated characteristic solution of B_L. We want to show that $\sigma_| = \tilde{\sigma}_|$. Diagram chasing shows that proving $r \circ P \circ o(\sigma) = i_* \circ P \circ o(\tilde{\sigma})$ should suffice. This follows from hypothesis 2, formulas (3.9) and from the fact that $P \circ o(\sigma) = s$ and $P \circ o(\tilde{\sigma}) = \tilde{s}$. $\square$

Let us now come back to the case of a Y^s-surgery. Let M be a closed oriented 3-manifold and let M_G be obtained from M by surgery along a Y-graph G. According to §3.2, $\mathcal{Y}$-surgery along G induces a bijective map:

$$Spin(M) \xrightarrow{\Theta_G} Spin(M_G).$$

With the notations of §3.3, each σ is sent by Θ_G to σ_G.

Suppose now that we are given a surgery presentation $M = V_L$ of M. Isotope the graph G in M to make it disjoint from the dual of L, so that G is in $\mathbf{S}^3 \setminus L$. Let H be a regular neighbourhood of G. A few Kirby's calculi, inside H, show that surgery along this Y-graph is equivalent to surgery on the two-component link K of Figure 3.1. We prefer this unsymmetric link to Figure 2.4 because of the fewer components.

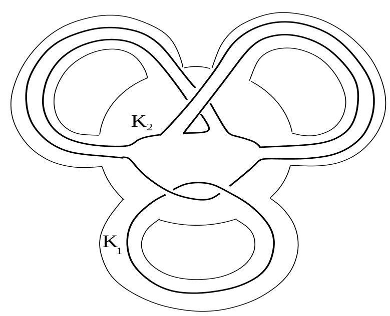

Figure 3.1: Y-surgery as surgery along a 2-component link.

We then have $M_G = V_{L\cup K}$. The linking matrix of $L \cup K$, when K is appropriately oriented, looks like:

$$B_{L\cup K} = \left(\begin{array}{ccc|cc} & & & x_1 & 0 \\ & B_L & & \vdots & \vdots \\ & & & x_l & 0 \\ \hline x_1 & \cdots & x_l & x & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right).$$

Writing the characteristic condition (3.7), we find that each characteristic solution of $B_{L\cup K}$ is of the form:

$$T_G(s) := \left(s_1, \dots, s_l, 0, x + \sum_{i=1}^{l} x_i s_i \right) \in (\mathbf{Z}_2)^{l+2}, \tag{3.10}$$

where $s = (s_1, \dots, s_l)$ must be characteristic for L. Equation (3.10) then defines a combinatorial bijection:

$$\mathcal{S}_L \xrightarrow[\simeq]{T_G} \mathcal{S}_{L\cup K}.$$

LEMMA 3.11. *With the above notations, the map T_G is a combinatorial version of the map Θ_G in terms of characteristic solutions for surgery presentations on $\mathbf{S}^3$. More precisely, the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{S}_L \arrow[r, "T_G"] & \mathcal{S}_{L\cup K} \\ Spin(M) \arrow[u, "\simeq"] \arrow[r, "\Theta_G"] & Spin(M_G) \arrow[u, "\simeq"'] \end{tikzcd}$$

PROOF. This follows from the definitions and from Lemma 3.10: note that K_2 is nul-homologous in H, and that here H_K is merely a handlebody. $\square$

DEFINITION 3.12. Let $M = V_L$ be a surgery presentation of a 3-manifold M on $\mathbf{S}^3$, and let G be a Y-graph in M. Then, G is said to be *simple* (with respect to this surgery presentation), if G can be isotoped in M so that, in $\mathbf{S}^3$, its leaves bound disjoint discs, each intersecting L in exactly one point.

COROLLARY 3.13. *For a Y^s-surgery along a simple Y-graph, the spin-diffeomorphism of Figure 3.2 holds.*

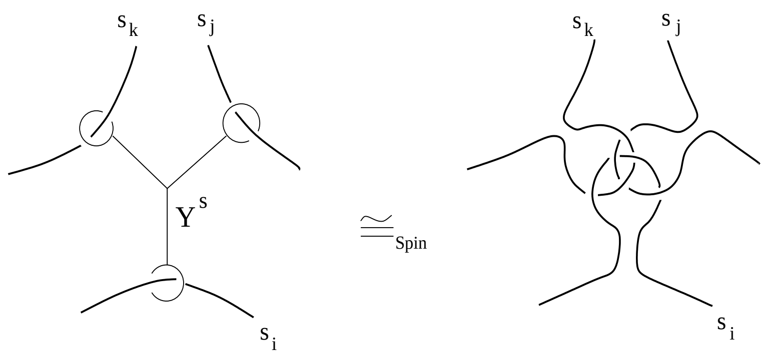

Figure 3.2: A simple Y^s-surgery.

PROOF. Replace in the lhs of Figure 3.2, this simple Y-graph by the 2-component link of Figure 3.1 such that K_1 links the ith component of L and use equation (3.10) to obtain the intermediate link of Figure 3.3. Perform then some spin Kirby's

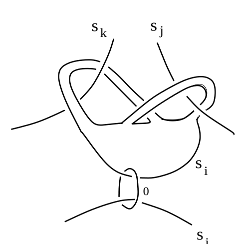

Figure 3.3: Some intermediate link and characterictic solution.

moves to obtain the rhs of Figure 3.2. $\square$

EXAMPLE 3.14. As a consequence of Corollary 3.13, the Lie spin structure of the 3-torus is Y^s-equivalent to the seven other ones.

PROOF. The two spin structures of $\mathbf{S}^1\times\mathbf{S}^2$ are equivalent, so are the eight ones of $\sharp^3\mathbf{S}^1\times\mathbf{S}^2$. Furthermore, $\sharp^3\mathbf{S}^1\times\mathbf{S}^2$ can be obtained from $\mathbf{S}^3$ by surgery along a trivial 0-framed three-component link. Surgery on the 0-framed Borromean rings gives rise to the 3-torus $\mathbf{S}^1\times\mathbf{S}^1\times\mathbf{S}^1$, and this link can be obtained from the trivial link by a simple Y-surgery. □

4. Quadratic forms on finite Abelian groups

4.1. Linking pairings, quadratic forms on finite Abelian groups and their presentations. We recall here standard algebraic constructions: notations are that of Deloup in [**De2**], where a brief review of the subject can be found.

DEFINITION 4.1. A *linking pairing* on a finite Abelian group G is a nondegenerate symmetric bilinear map $b : G\times G \longrightarrow \mathbf{Q}/\mathbf{Z}$.
A *quadratic form* on G is a map $q : G \longrightarrow \mathbf{Q}/\mathbf{Z}$ such that the map $b_q : G\times G \longrightarrow \mathbf{Q}/\mathbf{Z}$ defined by $b_q(x,y)=q(x+y)-q(x)-q(y)$ is bilinear, and such that q satisfies: $\forall x\in G,\ q(-x)=q(x)$. q is said to be *nondegenerate* when the associated bilinear form b_q is a linking pairing.

Let now (F,f) be a symmetric bilinear form on a free finitely generated Abelian group F. We denote by $ad_f : F \longrightarrow Hom(F,\mathbf{Z})$ the adjoint map, and by $a_f : F\otimes\mathbf{Q} \longrightarrow Hom(F,\mathbf{Q})$ its rational extension. Form:

$$K_f := \frac{Hom(F,\mathbf{Z})\cap Im(a_f)}{Im(ad_f)}.$$

Note that $K_f = T\left(Coker(ad_f)\right)$, the torsion subgroup of $Coker(ad_f)$. We now define a linking pairing:

$$K_f\times K_f \xrightarrow{L_f} \mathbf{Q}/\mathbf{Z}$$

by the formula:

$$L_f(\bar{x},\bar{y}) = x_{\mathbf{Q}}(\tilde{y}) \quad mod \quad 1, \tag{4.1}$$

where $x,y\in Hom(F,\mathbf{Z})\cap Im(a_f)$, $\tilde{y}\in F\otimes\mathbf{Q}$ is such that $a_f(\tilde{y})=y$, and $x_{\mathbf{Q}}\in Hom(F\otimes\mathbf{Q},\mathbf{Q})$ is the rational extension of x. (F,f) is said to be a *presentation* of the linking pairing (K_f,L_f).

Suppose now that the form (F,f) comes equipped with a *Wu class*, that is an element $w\in F$ such that $\forall x\in F, f(w,x)=f(x,x) \quad mod \quad 2$. We can then define a quadratic form over L_f, denoted by:

$$K_f \xrightarrow{\phi_{f,w}} \mathbf{Q}/\mathbf{Z}$$

and defined by:

$$\phi_{f,w}(\bar{x}) = \frac{1}{2}\left(x_{\mathbf{Q}}(\tilde{x}) - x(w)\right) \quad mod \quad 1. \tag{4.2}$$

$\phi_{f,w}$ is determined by the modulo $2F$ class of w. The triple (F,f,w) is said to be a *presentation* of the quadratic form $(K_f,\phi_{f,w})$.
Any linking pairing and any nondegenerate quadratic form admit such presentations with f nondegenerate (see [**W1**, Theorem (6)]).

Given an arbitrary quadratic form on a finite Abelian group (G,q), we can calculate its *Gauss sum*:

$$\gamma(G,q) = \frac{1}{\sqrt{|ker(b_q)|\cdot|G|}}\cdot\sum_{x\in G} e^{2i\pi q(x)} \in \mathbf{C}.$$

This complex number is an 8th root of unity or is 0 (only if q is degenerate).
We then define the corresponding *Gauss-Brown invariant* $B(G,q) \in \overline{\mathbf{Z}_8} = \mathbf{Z}_8 \cup \{\infty\}$ by the formula:

$$\gamma(G,q) = e^{\frac{2i\pi}{8}\cdot B(G,q)} \in \mathbf{C}, \tag{4.3}$$

using the convention $e^{2i\pi\cdot\infty} = 0$.
If (G,q) admits a triple (F,f,w) as a presentation, a useful formula of Van der Blij states that[2]:

$$B(K_f, \phi_{f,w}) = sgn(f) - f(w,w) \quad mod \quad 8. \tag{4.4}$$

For (G,b) a linking pairing, denote by $Quad(G,b)$ the set of quadratic forms with b as associated linking pairing, and denote by $T_2(G)$ the subgroup of elements of G of order at most 2. Using the nondegenerativity of b, we easily obtain:

LEMMA 4.2. *The set $Quad(G,b)$ is an affine space over $T_2(G)$, with action defined by:*

$$\forall x \in T_2(G),\ \forall q \in Quad(G,b),\ x\cdot q := q + b(x,-).$$

The following lemma says how the Gauss-Brown invariant behaves under this action.

LEMMA 4.3. *Let $q \in Quad(G,b)$ and let $x \in T_2(G)$. Then,*

$$\gamma(G, x\cdot q) = e^{-2i\pi q(x)} \cdot \gamma(G,q).$$

PROOF. Since $(x\cdot q)(y) = q(y) + b(x,y) = q(x+y) - q(x)$, we have:

$$\begin{array}{rcl} \sqrt{|G|}\cdot\gamma(G,x\cdot q) & = & \sum_{y\in G} e^{2i\pi(x\cdot q)(y)} \\ & = & e^{-2i\pi q(x)} \cdot \sum_{y\in G} e^{2i\pi q(x+y)} \\ & = & e^{-2i\pi q(x)} \cdot \sqrt{|G|}\cdot\gamma(G,q) \end{array}$$

□

4.2. Isomorphism classes of nondegenerate quadratic forms. We now want to prove the following result:

THEOREM 4.4. *Let (G,q) and (G',q') be two nondegenerate quadratic forms on finite Abelian groups. We denote by b and b' the linking pairings going respectively with them. The following two assertions are equivalent:*

1. $(G,q) \simeq (G',q')$,
2. $(G,b) \simeq (G',b')$ *and* $B(q) = B(q') \in \mathbf{Z}_8$,

where $B(-)$ denotes the Gauss-Brown invariant of quadratic forms.

Let (G,q) (respectively (G',q')) be a nondegenerate quadratic form over the linking pairing (G,b) (respectively (G',b')). Recall that the relation between b and q is:

$$b(x,y) = q(x+y) - q(x) - q(y). \tag{4.5}$$

From this formula and the definition of $B(-)$, "$1 \Rightarrow 2$" of Theorem 4.4 is obvious.

Suppose momentarily that G is a p-group, with p an odd prime. Then, equation $2q(x) = b(x,x)$ makes b determine q, for if q'' is another quadratic form over b, then $(q - q'')$ is an order at most 2 element of $Hom\,(G, \mathbf{Q}/\mathbf{Z}) \simeq G$, and so vanishes. So, if G and G' are both p-groups, then $b \simeq b'$ implies $q \simeq q'$.

[2] A sketch of proof of this formula can be found in [**VdB**], in case when f is nondegenerate. Milnor and Husemoller have included a detailed proof in [**MH**, Appendix 4], for f nondegenerate and $w = 0$. The general case can be reduced to this special case.

Come back now to the general case and suppose that condition 2 is satisfied. (G,b) splits along its p-primary components G_p:

$$(G,b) = \bigoplus_{p,\text{ prime}} \left(G_p, b|_{G_p\times G_p}\right),$$

and, according to formula (4.5), the same holds for q. The given isomorphism between (G,b) and (G',b'), induces then for each prime p an isomorphism between $\left(G_p, b|_{G_p\times G_p}\right)$ and $\left(G'_p, b'|_{G'_p\times G'_p}\right)$. From the above lines, we deduce that, for p odd, $\left(G_p, q|_{G_p}\right)$ and $\left(G'_p, q'|_{G'_p}\right)$ are isomorphic.
In particular, $B\left(q|_{G_p}\right) = B\left(q'|_{G'_p}\right)$ for p odd, and so, by additivity of the Gauss-Brown invariant, this is also true for $p=2$. Consequently, it is enough to prove Theorem 4.4 when G and G' are 2-groups.

In the sequel, we recall a construction due to Wall (see [**W1**, §6]), establishing a one to one correspondence (up to isomorphism) between nondegenerate quadratic forms on 2-groups and linking pairings on 2-groups without direct summand of order two. Next, we give a brief review of Kawauchi and Kojima classification of linking pairings on 2-groups. We will finally end the proof of Theorem 4.4.

Wall's construction. A linking pairing (G',b') will be said here to be *special* if G' is a finite 2-group without direct summand of order two.
Let give us a special linking pairing (G',b'). Set $G := G'/T_2(G')$ where $T_2(G')$ is the subgroup of elements of order at most 2, and denote by π the canonical projection $G' \longrightarrow G$. Note that G can be any 2-group. Define now $q : G \longrightarrow \mathbf{Q}/\mathbf{Z}$ by:

$$\forall x = \pi(x') \in G, \quad q(x) = b'(x',x').$$

The quantity $q(x)$ is well-defined because of the special feature of the group G'.
Then, q is easily seen to be quadratic and nondegenerate, with associated linking pairing $b : G\times G \longrightarrow \mathbf{Q}/\mathbf{Z}$ defined by:

$$\forall x = \pi(x'),\ \forall y = \pi(y') \in G,\ b(x,y) = 2\cdot b'(x',y'). \tag{4.6}$$

Let us denote by Ψ the construction $(G',b') \longmapsto (G,q)$. Wall showed this to be surjective onto the set of nondegenerate quadratic forms on 2-groups. He also proved that if (G',b'_1) and (G',b'_2) give rise to the same (G,q) by Ψ, then they have to be isomorphic (see [**W1**, Theorem 5]).

As a consequence, the classification, up to isomorphism, of nondegenerate quadratic forms on 2-groups is reduced to that of special linking pairings.

Kawauchi-Kojima classification of linking pairings on 2-groups. Let (G,b) be a linking pairing on a finite 2-group. Find a cyclic decomposition of G:

$$G \simeq \bigoplus_{k\geq 1} \left(\mathbf{Z}_{2^k}\right)^{r_k}.$$

The natural numbers $r_k = r_k(b)$ are group invariants of G. The very next construction is due to Wall (see [**W1**, §5]).

Denote by $\overline{G}_k$ the subgroup of G of elements of order at most 2^i for $i\leq k$, and set:

$$\tilde{G}_k = \frac{\overline{G}_k}{\overline{G}_{k-1} + 2\cdot \overline{G}_{k+1}}.$$

The group $\tilde{G}_k$ is clearly a $\mathbf{Z}_2$-vector space of rank r_k. Let also $\tilde{b}_k : \tilde{G}_k\times\tilde{G}_k \longrightarrow \mathbf{Q}/\mathbf{Z}$ be defined by:

$$\forall x,y \in \overline{G}_k, \quad \tilde{b}_k(\tilde{x},\tilde{y}) = 2^{k-1}\cdot b(x,y).$$

The form $\tilde{b}_k$ was shown by Wall to be nondegenerate (see [**W1**, Lemma 8]).

Consider now the map $\tilde{G}_k \longrightarrow \mathbf{Q}/\mathbf{Z}$ sending $\tilde{x}$ to $\tilde{b}_k(\tilde{x},\tilde{x})$. It is additive, and so we can define an element $\tilde{c}_k = \tilde{c}_k(b)$ in $\tilde{G}_k$ by the equation:

$$\forall \tilde{x} \in \tilde{G}_k, \quad \tilde{b}_k(\tilde{x},\tilde{c}_k) = \tilde{b}_k(\tilde{x},\tilde{x}). \tag{4.7}$$

When $\tilde{c}_k = 0$, the following map can be defined:

$$\left\{ \begin{array}{ccc} \frac{G}{\overline{G}_k} & \xrightarrow{q_k} & \frac{\mathbf{Q}}{\mathbf{Z}} \\ \bar{x} & \longmapsto & 2^{k-1} \cdot b(x,x). \end{array} \right.$$

The form $q_k = q_k(b)$ can be verified to be quadratic and nondegenerate. In particular, its Gauss-Brown invariant $B(q_k)$ is not equal to ∞.
Kawauchi and Kojima defined $\sigma_k = \sigma_k(b) \in \overline{\mathbf{Z}_8}$ by:

$$\sigma_k = \left\{ \begin{array}{ll} B(q_k) & \text{if } \tilde{c}_k = 0, \\ \infty & \text{otherwise,} \end{array} \right.$$

and showed the following theorem (see [**KK**, Theorem 4.1]).

THEOREM 4.5 (Kawauchi-Kojima). *If (G,b) is a linking pairing on a finite 2-group, its isomorphism class is determined by the invariant family $(r_k(b),\sigma_k(b))_{k\geq 1}$.*

END OF PROOF OF THEOREM 4.4. Let (G,q) be a quadratic form on a finite 2-group G, going with a linking pairing b. Let also (G',b') be a special linking pairing, giving rise to (G,q) by Wall's construction Ψ.

We want to compare the invariants r_k and σ_k of b and b', in order to quantify how much (G,b) determines (G',b'), and so (G,q), up to isomorphism.
Recall that $G = G'/\overline{G'}_1$. Denote by $\pi : G' \longrightarrow G$ the canonical projection. The map π induces a morphism from $\overline{G'}_{k+1}$ onto $\overline{G}_k$ with kernel $\overline{G'}_1$. So, π induces a natural isomorphism:

$$\tilde{G'}_{k+1} \xrightarrow[\simeq]{\tilde{\pi}_k} \tilde{G}_k.$$

In particular, the $\mathbf{Z}_2$-vector spaces $\tilde{G'}_{k+1}$ and $\tilde{G}_k$ have the same rank. So,

$$\forall k \geq 1, \quad r_{k+1}(G') = r_k(G). \tag{4.8}$$

Besides, since G' is special, we have:

$$r_1(G') = 0. \tag{4.9}$$

The isomorphism $\tilde{\pi}_k$ makes $\tilde{b'}_{k+1}$ and $\tilde{b}_k$ commute because of equation (4.6). As a consequence, $\tilde{\pi}_k$ sends $\tilde{c}_{k+1}(b')$ to $\tilde{c}_k(b)$. Furthermore, when these (simultaneously) vanish, the natural isomorphism between $G'/\overline{G'}_{k+1}$ and $G/\overline{G}_k$ induced by π, make $q_{k+1}(b')$ and $q_k(b)$ commute (because of equation (4.6)). As a consequence, these two quadratic forms will have the same Gauss-Brown invariant. So, to sum up,

$$\forall k \geq 1, \quad \sigma_{k+1}(b') = \sigma_k(b) \in \overline{\mathbf{Z}_8}. \tag{4.10}$$

Since $\tilde{G'}_1 = 0$, $\tilde{c}_1(b')$ vanishes. It remains to be noticed that $q_1(b')$ is nothing but q. Thus,

$$\sigma_1(b') = B(q). \tag{4.11}$$

Now, from equations (4.8), (4.9), (4.10), (4.11) and Kawauchi-Kojima theorem, we see that (G,b) together with $B(q)$ determine (G,q) up to isomorphism. What has been remaining to be proved for Theorem 4.4, then follows. □

We now give a result of Durfee (see [**Du**, Corollary 3.9]) as a corollary of Theorem 4.4.

COROLLARY 4.6. *Let $b : G \times G \longrightarrow \mathbf{Q}/\mathbf{Z}$ be a linking pairing on a finite Abelian group G* without cyclic direct summand of order 2 or 4. *Then,* $\forall q, q' \in Quad(G, b)$, $q \simeq q'$.

PROOF. Take some quadratic forms q and q' over b, and let $x \in T_2(G)$ be such that $q' = x \cdot q$ (see Lemma 4.2). By the hypothesis on G, there exits some $x_0 \in G$ such that $x = 4x_0$, and so, by homogeneity of q, we have:

$$q(x) = q(4x_0) = 4^2 q(x_0).$$

But since x_0 is then of order at most 8, $q(x_0)$ has to be of order at most $2 \cdot 8 = 16$ (see [**De1**, Lemma 1.12]). It follows that $q(x) = 0$ and so, by Lemma 4.3, we obtain that $B(G, q) = B(G, q')$. Theorem 4.4 allows us to conclude. $\square$

5. The quadratic form $\phi_{M,\sigma}$

In this section, when not specified, integer coefficients are assumed.

5.1. Turaev 4-dimensional definition of $\phi_{M,\sigma}$. Let M be a connected closed oriented 3-manifold, and let $\psi : V_L \longrightarrow M$ be a surgery presentation on $\mathbf{S}^3$ given by an ordered oriented framed link L (see the beginning of §3.6).
We use notations and apply constructions of §4.1 to $F = H_2(W_L)$, taking for f the intersection form of W_L. Recall that the matrix of f relative to the preferred basis of F is B_L, the linking matrix of L.
The composite:

$$H_2(W_L) \xrightarrow{ad_f} Hom(H_2(W_L), \mathbf{Z}) \xleftarrow[\simeq]{} H^2(W_L) \xrightarrow[\simeq]{P} H_2(W_L, \partial W_L)$$

is equal to $i_* : H_2(W_L) \longrightarrow H_2(W_L, \partial W_L)$, induced by inclusion. Since $Coker(i_*) = H_1(V_L)$, we obtain the following isomorphism r:

$$\begin{tikzcd} K_f \arrow[r, "\simeq"'] \arrow[rr, dashed, bend right=60, "r"'] & T(Coker(i_*)) \arrow[r, "\psi_*", "\simeq"'] & TH_1(M) \end{tikzcd}$$

In fact, it is well-known that the above $(F, -f)$ is *via* r a presentation of the torsion linking form $\big(TH_1(M), \lambda_M\big)$ of M, the definition of which we now recall:

> Let $x, x' \in TH_1(M)$ be respectively realized by oriented disjoint knots K, K' in M. Let $c' \in \mathbf{N}$ be such that $c' \cdot x' = 0$. Pick a c'-times connected sum of K'. We obtain a null-homologous knot in M for which we can thus find a Seifert surface S' in general position with K. Then:
>
> $$\lambda_M(x, x') = \frac{1}{c'} K \bullet S' \in \mathbf{Q}/\mathbf{Z},$$
>
> where $\bullet$ is the intersection form of M.

Now to each $\sigma \in Spin(M)$ is associated a characteristic solution of $-B_L$ or, alternatively, a Wu class (modulo $2F$) of $-f$, denoted by w_σ. Then, Turaev defined:

DEFINITION 5.1. The *quadratic form of the spin 3-manifold* (M, σ):

$$TH_1(M) \xrightarrow{\phi_{M,\sigma}} \mathbf{Q}/\mathbf{Z}$$

is defined to be $\phi_{-f, w_\sigma} \circ r^{-1}$.

We still have to verify that $\phi_{M,\sigma}$ does not depend on the choice of the surgery presentation.
Let $\psi' : V_{L'} \longrightarrow M$ be another one. Let $s \in (\mathbf{Z}_2)^l$ (resp. s') be the characteristic solution of B_L (resp. $B_{L'}$) corresponding to the spin structure $\psi^*(\sigma)$ of V_L (resp. $(\psi')^*(\sigma)$ of $V_{L'}$).
According to the refined Kirby's theorem (see [**Bl**]), there exists a sequence of spin Kirby's moves from (L,s) to (L',s'), inducing a spin-diffeomorphism from $(V_L, \psi^*(\sigma))$ to $(V_{L'}, (\psi')^*(\sigma))$ isotopic to $(\psi')^{-1} \circ \psi$. These Kirby's moves induce a path $(F, f, w_\sigma) \rightsquigarrow (F', f', w'_\sigma)$ whose elementary steps are:

$$\left\{ \begin{array}{lll} (F,f,w) \longmapsto \left(F, {}^t SfS, S^{-1}(w)\right) & & \text{with } S \in Aut(F), \\ (F,f,w) \longmapsto (F \oplus \mathbf{Z}, f \oplus (\pm 1), w \oplus (1)) \, . & & \end{array} \right.$$

As a consequence, this path induces an isomorphism t from (K_f, ϕ_{f,w_σ}) to $(K_{f'}, \phi_{f',w'_\sigma})$ making the following diagram commutative:

$$\begin{tikzcd} K_f \arrow[r, "t"] \arrow[dr, "r"'] & K_{f'} \arrow[d, "r'"] \\ & TH_1(M) \end{tikzcd}$$

The well-definition of $\phi_{M,\sigma}$ then follows.

5.2. An intrinsic definition for $\phi_{M,\sigma}$. Let (M,σ) be a closed spin 3-manifold and let K be a smooth oriented knot in M.
Each parallel l of K defines a trivialization of the normal bundle of K in M, and so allows us to restrict σ to a spin structure on $K \cong \mathbf{S}^1$. We define:

$$\sigma(K,l) = \left\{ \begin{array}{ll} 0 & \text{if the induced spin structure on } \mathbf{S}^1 \text{ is the "bounding" one,} \\ 1 & \text{otherwise.} \end{array} \right.$$

$\sigma(K,l)$ is a $\mathbf{Z}_2$-valued invariant of framed knots in M.

LEMMA 5.2. *Let (M,σ) be a closed spin 3-manifold. Then, for each oriented smooth knot K in M with l as a parallel and meridian μ,*

$$\sigma(K, l + \mu) = \sigma(K,l) + 1 \in \mathbf{Z}_2 .$$

PROOF. Let S denote the boundary of a regular neighbourhood of K in M. The normal bundle of S in M is naturally trivialized, so S inherits from (M,σ) a spin structure. Let q be the quadratic form associated to the spin smooth surface $(S, \sigma|_S)$ as defined by Johnson in [**J1**]. The following identity then holds for each parallel l:

$$\sigma(K,l) = q\left([l]\right),$$

when l is thought of as a curve on S. Since q is quadratic with respect to the modulo 2 intersection form $\bullet$ on S, we have:

$$\begin{array}{rcl} \sigma(K, l+\mu) & = & q\left([l] + [\mu]\right) \\ & = & q\left([l]\right) + q\left([\mu]\right) + [l] \bullet [\mu] \\ & = & q\left([l]\right) + 1 \\ & = & \sigma(K,l) + 1. \end{array}$$

□

We now recall the definition of the *framing number* $Fr(K,l) \in \mathbf{Q}$ of a rationally nulhomologous oriented framed knot (K,l) in a closed oriented 3-manifold M:

Choose $c \in \mathbf{N}^*$ such that $c \cdot [K] = 0 \in H_1(M)$. Pick a c-times connected sum of K. We obtain a null-homologous knot in M for which we can thus pick a Seifert surface S in general position with the knot l. Then:

$$Fr(K,l) = \frac{1}{c} \cdot l \bullet S \in \mathbf{Q}.$$

LEMMA 5.3. *Let (M,σ) be a closed spin 3-manifold and $x \in TH_1(M)$. Choose a smooth oriented knot K in M representative for x, and pick a parallel l for K satisfying $\sigma(K,l) = 0 \in \mathbf{Z}_2$. Then,*

$$\phi_{M,\sigma}(x) = \frac{1}{2} \cdot Fr(K,l) \in \frac{\mathbf{Q}}{\mathbf{Z}}. \tag{5.1}$$

Note that, according to Lemma 5.2, the rhs of (5.1) is an invariant of the oriented knot K (it does not depend on the choice of l satisfying the above condition). This lemma claims that it only depends on the homology class x of K, and *gives a 3-dimensional definition for the quadratic form $\phi_{M,\sigma}$.*

REMARK 5.4. From this lemma, we can see that $\phi_{M,\sigma}$ coincides with the quadratic form defined by Lannes and Latour in [**LL**] when specialized to our case (see also [**MS**]).

PROOF OF LEMMA 5.3. Consider the 4-manifold W_1 obtained from $M \times [0,1]$ by attaching a 2-handle to $M \times 1$ along (K,l). Identify M with $M \times 0$. Since $\sigma(K,l) = 0$, σ extends in a unique way to a spin structure σ_1 of W_1.
(W_1,σ_1) is then a spin cobordism between (M,σ) and $(-M',-\sigma')$, where M' is the closed oriented 3-manifold obtained from M by the corresponding surgery, and where $-\sigma'$ is the restriction of σ_1 to $-M'$.
Note also that the core of the 2-handle is a 2-disc D in W_1 with boundary K in M, and whose normal bundle can be trivialized in accordance with the trivialization of the normal bundle of K in M given by l. The framed knot (K,l) will briefly be said to *have property* $(\mathcal{D})$ in W_1.
According to Kaplan Theorem (see [**Ka**]), the spin 3-manifold (M',σ') admits an *even* surgery presentation in $\mathbf{S}^3$ (*i.e.* the linking matrix is even and its characteristic solution corresponding to σ' is the trivial one). Denote by W_2 the trace of the surgery and by σ_2 the unique extension of σ' to the whole of W_2.
By gluing (W_1,σ_1) to (W_2,σ_2) along (M',σ'), we obtain a spin 4-manifold (W,σ) with boundary (M,σ). The 2-handle from M to M' can be reversed. After a rearrangement, the 4-manifold W appears as $\mathbf{B}^4$ to which have been simultaneously attached some 2-handles (one more than W_2), with boundary M, and to which σ can be extended. So, W is the trace of an even surgery presentation.

As a summary, we have found so far an even surgery presentation $(L,0)$ for (M,σ) such that (K,l) has property $(\mathcal{D})$ in W_L.

Let us work with this surgery presentation of (M,σ). Notations of §5.1 will be used: $F = H_2(W_L)$, f stands for the intersection form of W_L and so on.
Let the 2-disc D give an element of $H_2(W_L,\partial W_L)$. The latter is identified with an element d of $Hom(F,\mathbf{Z})$. Recall from the definition of $\phi_{M,\sigma}$ that, in this even case,

$$\phi_{M,\sigma}(x) = -\frac{1}{2} d_{\mathbf{Q}}(\tilde{d}), \tag{5.2}$$

where $d_{\mathbf{Q}}$ is the rational extension of d and where $\tilde{d} \in F \otimes \mathbf{Q}$ is such that $a_f(\tilde{d}) = d$.
Let $c \in \mathbf{N}$ such that $c \cdot x = 0 \in H_1(M)$. Then, there exists $y \in F$ such that $ad_f(y) = c \cdot d$. So $\frac{1}{c} \cdot y \in F \otimes \mathbf{Q}$ works as a $\tilde{d}$. Equation (5.2) can be rewritten as:

$$\phi_{M,\sigma}(x) = -\frac{1}{2c^2} f(y,y). \tag{5.3}$$

When y is seen as belonging to $H_2(W_L)$, the integer $f(y,y)$ is equal to $Y \bullet Y'$, where Y and Y' are 2-cycles representatives for y in transverse position in W_L. By means of a "collar" trick appearing in [**De1**], we will be able to give examples of such Y and Y'.
We add a collar $M \times [0,1]$ to W_L such that $M \times 0$ is identified with M. Let S be a Seifert surface for $c \cdot K$ in M in transverse position with l, and S' be a Seifert surface for $(c \cdot l) \times 1$ in $M \times 1$. Because of the property $(\mathcal{D})$, D can be pushed off to a disc D' in such a way that $\partial D' = l$ and $D \cap D' = \varnothing$. Figure 5.1 is a good summary. We define $Y = c.D - S$ and $Y' = (c.D' + (c.l) \times [0,1]) - S'$. Then:

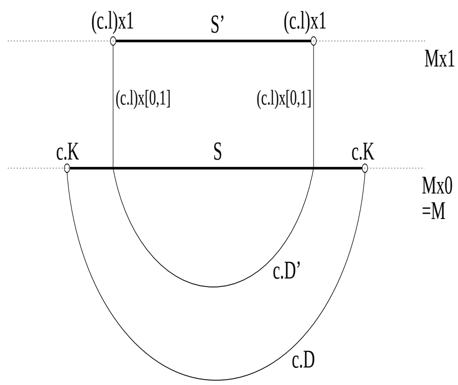

Figure 5.1: Calculating $Y \bullet Y'$.

$$f(y,y) = Y \bullet Y' = (-S) \bullet (c.D') = -c \cdot S \bullet l. \tag{5.4}$$

The last $\bullet$ in (5.4) is intersection in M. The lemma follows from (5.4), (5.3) and the definition of a framing number. $\square$

5.3. Properties of $\phi_{M,\sigma}$. The algebraic results of §4 have a topological meaning. First, it has been shown by Turaev in [**T2**, Theorem V]:

LEMMA 5.5 (Turaev). *For each $\sigma \in Spin(M)$, $B(\phi_{M,\sigma}) = -R_M(\sigma) \mod 8$, where R_M is the Rochlin function of the 3-manifold M.*

PROOF. Find an *even* surgery presentation of the spin 3-manifold (M,σ). So we are led to apply formula (4.4) with $w = 0$. $\square$

So, in view of Lemma 5.5, the topological translations of Theorem 4.4 and its Corollary 4.6 are respectively:

PROPOSITION 5.6. *Let (M,σ) and (M',σ') be connected closed spin 3-manifolds. The following two assertions are equivalent:*

1. *their quadratic forms $\phi_{M,\sigma}$ and $\phi_{M',\sigma'}$ are isomorphic,*
2. *their linking forms λ_M and $\lambda_{M'}$ are isomorphic, and $R_M(\sigma) = R_{M'}(\sigma')$ modulo 8.*

COROLLARY 5.7. *Let M be a connected closed oriented 3-manifold, such that $H_1(M)$ does not admit $\mathbf{Z}_2$ nor $\mathbf{Z}_4$ as a direct summand. Then, all of its quadratic forms are isomorphic one to another.*

6. Proof of refined Matveev theorem

Part of the work has already been done in previous sections. First, "$1 \Longrightarrow 3$" follows from Corollary 3.8 and from the easy part of (unspun) Matveev theorem: a Y-surgery preserves homology and torsion linking forms. This can be verified seeing Y-surgery as a $\mathcal{V}_h$-surgery (where h is the Borromean diffeomorphism of Remark 2.8), using a Mayer-Vietoris argument and the fact that h belongs to the Torelli group. Second, "$3 \Longrightarrow 2$" follows from Proposition 5.6. What remains to be proved is then "$2 \Longrightarrow 1$".

We start by recalling an algebraic result of Durfee (see [**Du**]) about even symmetric bilinear forms on finitely generated free Abelian groups.
Let $(\mathbf{Z}^2, h)$ and $(\mathbf{Z}^8, \gamma_8)$ be the unimodular even forms whose matrices are respectively:

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma_8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{pmatrix}.$$

DEFINITION 6.1. Let (F_1, f_1) and (F_2, f_2) be two symmetric even bilinear forms on finitely generated free Abelian groups. They are said to be *stably equivalent* if they become isomorphic after some stabilizations with unimodular symmetric even bilinear forms.

Note that a unimodular even form becomes indefinite after taking direct sum with h. Recall that every even unimodular indefinite form splits as a direct sum of h and γ_8 (see for example [**Ki**, p.26]). Thus, in Definition 6.1, h and γ_8 suffice as unimodular even forms to stabilize.

For (F, f) an even symmetric bilinear form on a finitely generated free Abelian group, we will shortly denote by ϕ_f the quadratic form $\phi_{f,0}$ corresponding to the zero Wu class.

PROPOSITION 6.2 (Durfee). *Let (F_1, f_1) and (F_2, f_2) be two symmetric even bilinear forms on finitely generated free Abelian groups. Then, the following two assertions are equivalent:*

(1) *(F_1, f_1) and (F_2, f_2) are stably equivalent,*
(2) *$Ker\,(ad_{f_1}) \simeq Ker\,(ad_{f_2})$ and $\phi_{f_1} \simeq \phi_{f_2}$.*

PROOF. Implication "1⇒2" is obvious since γ_8 and h are both unimodular. Now suppose that condition 2 is satisfied. For each $i \in \{1, 2\}$, there exists a nondegenerate symmetric bilinear form $\left(\tilde{F}_i, \tilde{f}_i\right)$ such that:

$$(F_i, f_i) \simeq \left(\tilde{F}_i, \tilde{f}_i\right) \oplus (\mathbf{Z}^r, O_r)$$

where $\tilde{F}_i = F_i / Ker\,(ad_{f_i})$ (note that it is free), $r = rk\,(Ker\,(ad_{f_1})) = rk\,(Ker\,(ad_{f_2}))$, and O_r is the zero form. The form $\tilde{f}_i$ is still even and $\phi_{\tilde{f}_1}$ and $\phi_{\tilde{f}_2}$ are still isomorphic.
Consequently, without loss of generality, we can assume both f_i to be nondegenerate. But this case was treated by Durfee in [**Du**, Corollary 4.2.(ii)][3]. □

[3] An easier and direct proof of this result was given by Wall in [**W2**, Corollary 1].

Let (M,σ) and (M',σ') be connected closed spin 3-manifolds such that $\beta_1(M) = \beta_1(M')$ and $\phi_{M,\sigma} \simeq \phi_{M',\sigma'}$.
Suppose given for them some even surgery presentations $(L,0)$ and $(L',0)$ with respective linking matrices B and B'. According to Proposition 6.2, there exists a unimodular integer matrix P satisfying for some stabilizations:

$$^tP \cdot (B \oplus H \cdots H \oplus \Gamma_8 \cdots \Gamma_8) \cdot P = B' \oplus H \cdots H \oplus \Gamma_8 \cdots \Gamma_8.$$

We have the following geometric realizations of algebraic operations:

(1) stabilizations by H correspond to connected sums with $\mathbf{S}^3$ surgery presented on the zero-framed Hopf link,
(2) a stabilization by Γ_8 is concrete when thought of as a connected sum with the Poincaré sphere surgery presented on an appropriate height-component link as in [**Ki**, Figure 5.3, p.15],
(3) congruence by P can be realized by some spin Kirby's moves (handle-slidings and changes of orientation of components of L).

The Poincaré sphere can also be obtained by surgery along a $(+1)$-framed trefoil knot ([**Ki**, Figure 5.3, p.15]), which can be obtained from the $(+1)$-framed unknot by a simple Y-surgery (see Definition 3.12). As a consequence, the Poincaré sphere and the sphere $\mathbf{S}^3$, equipped with their unique spin structures, are Y^s-equivalent. Since Y^s-equivalence is compatible with connected sums, *we can assume* that $B = B'$.

A theorem of Murakami and Nakanishi ([**MN**, Theorem 1.1][4]) states that two ordered oriented links have identical linking matrices if and only if they are Δ-equivalent. A Δ-move is a certain unknotting operation, which is equivalent to surgery along a simple Y-graph.
Finally, from Corollary 3.13, we see that a simple Y^s-surgery between even surgery presentations leaves the trivial characteristic solution fixed. We conclude that (M,σ) is Y^s-equivalent to (M',σ'), which completes the proof.

7. Applications

According to Theorem 1.1, two connected closed spin 3-manifolds are Y^s-equivalent if and only if they are Y-equivalent as *plain* 3-manifolds and their Rochlin invariants are identical modulo 8. In other words, while studying the degree 0 part of Goussarov-Habiro theory, the spin problem can be "factored out".

Now, given a closed connected oriented 3-manifold, one can wonder whether all of its spin structures are Y^s-equivalent one to another. This has been verified to be true in the case of $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ by a direct calculation (Example 3.14). More generally we have:

COROLLARY 7.1. *Let M be a connected oriented closed 3-manifold such that $H_1(M)$ has no cyclic direct summand of order 2 or 4. Then, all spin structures of M are Y^s-equivalent one to another.*

PROOF. This follows directly from Theorem 1.1 and Corollary 5.7. □

On the contrary, we have:

EXAMPLE 7.2. The two spin structures of $\mathbf{RP}^3$ are not Y^s-equivalent, for the Rochlin function of $\mathbf{RP}^3$ takes 1 and -1 as values.

[4] In fact, the first reference is Matveev, but the proof in [**Mt**] is not detailed.

CHAPITRE 3

Invariants de type fini des cylindres d'homologie

En collaboration avec Jean-Baptiste Meilhan.

1. Introduction

1.1. Homology cylinders. Homology cylinders are important objects in the theory of finite type invariants of Goussarov-Habiro: they have thus appeared in both [**Hr**] and [**Go**]. Let us recall the definition of these objects.

Let Σ be a compact connected oriented surface. A *homology cobordism* over Σ is a triple (M, i^+, i^-) where M is a compact oriented 3-manifold and $i^\pm : \Sigma \longrightarrow M$ are oriented embeddings with images $\Sigma^\pm$, such that:

(i) $i^\pm$ are homology isomorphisms;
(ii) $\partial M = \Sigma^+ \cup (-\Sigma^-)$ and $\Sigma^+ \cap (-\Sigma^-) = \pm\partial\Sigma^\pm$;
(iii) $i^+|_{\partial\Sigma} = i^-|_{\partial\Sigma}$.

Homology cobordisms are considered up to orientation-preserving diffeomorphisms. When $(i^-)_*^{-1} \circ (i^+)_* : H_1(\Sigma; \mathbf{Z}) \longrightarrow H_1(\Sigma; \mathbf{Z})$ is the identity, M is said to be a *homology cylinder*. The set of homology cobordisms is denoted here by $\mathcal{C}(\Sigma)$, and $\mathcal{HC}(\Sigma)$ denotes the subset of homology cylinders. If $M = (M, i^+, i^-)$ and $N = (N, j^+, j^-)$ are homology cobordisms, we can define their *stacking product* by

$$M \cdot N := (M \cup_{i^- \circ (j^+)^{-1}} N, i^+, j^-).$$

This product induces a monoid structure on $\mathcal{C}(\Sigma)$, with $\mathcal{HC}(\Sigma)$ a submonoid. The unit element is $1_\Sigma := (\Sigma \times I, Id, Id)$, where I is the unit interval $[0,1]$ and where a collar of $\Sigma^\pm$ is stretched along $\partial\Sigma \times I$ so that the second defining condition for homology cobordisms is satisfied.

Habiro in [**Hr**, §8.5] outlined how homology cylinders can serve as a powerful tool in studying the mapping class groups of surfaces (see [**GL**], [**Hg**], [**Le**]). The connection lies on the homomorphism of monoids

$$\mathcal{T}(\Sigma) \xrightarrow{C} \mathcal{HC}(\Sigma)$$

sending each h in the Torelli group of Σ to the mapping cylinder $C_h = (\Sigma \times I, Id, h)$ (with, as above, a collar of $\Sigma^\pm$ stretched along $\partial\Sigma \times I$).
In the sequel, *we restrict ourselves* to the following two cases:

(i) $\Sigma = \Sigma_g$ is the standard closed oriented surface of genus $g \geq 0$, which here is referred to as the *closed case*;
(ii) $\Sigma = \Sigma_{g,1}$ is the standard compact oriented surface of genus $g \geq 0$ with one boundary component, which here is referred to as the *boundary case*.

The usual notations $\mathcal{T}_{g,1} = \mathcal{T}(\Sigma_{g,1})$ and $\mathcal{T}_g = \mathcal{T}(\Sigma_g)$ for the Torelli groups will be used. Also denote by H the first homology group of Σ with integer coefficients, by $\bullet$ the intersection form on H and by $(x_i, y_i)_{i=1}^g$ a symplectic basis for $(H, \bullet)$.

1.2. Y_k-equivalence. The theory of finite type invariants of Goussarov-Habiro has come equipped with a topological calculus toolbox: this was called *calculus of claspers* in [**Hr**] or alternatively *clovers* in [**GGP**]. We will assume a certain familiarity of the reader with these techniques.
In particular, let us recall that, for $k \geq 1$ an integer, the *Y_k-equivalence*[1] is the equivalence relation generated by surgery on connected clovers of degree k. Following Habiro in [**Hr**], we can then define a descending filtration of monoids

$$\mathcal{C}(\Sigma) \supset \mathcal{C}_1(\Sigma) \supset \mathcal{C}_2(\Sigma) \supset \cdots \supset \mathcal{C}_k(\Sigma) \supset \cdots$$

where $\mathcal{C}_k(\Sigma)$ is the submonoid consisting of the homology cobordisms which are Y_k-equivalent to the trivial cobordism 1_Σ. Note the following fact, a proof of which has been inserted in §4.

Proposition 1.1. *If $\Sigma = \Sigma_g$ or $\Sigma_{g,1}$, then $\mathcal{HC}(\Sigma) = \mathcal{C}_1(\Sigma)$.*

As mentioned by Habiro, we can show from the calculus of clovers that for every $k \geq 1$, the quotient monoid

$$\overline{\mathcal{C}}_k(\Sigma) := \mathcal{C}_k(\Sigma)/Y_{k+1}$$

is an Abelian group. In particular, $\overline{\mathcal{C}}_1(\Sigma)$ is the Abelian group of homology cylinders over Σ up to Y_2-equivalence. This group is the subject of the present paper.

For $k \geq 2$, Habiro gives a combinatorial upper bound for the Abelian group $\overline{\mathcal{C}}_k(\Sigma)$. Precisely, he defines $\mathcal{A}_k(H)$ to be the Abelian group (finitely) generated by unitrivalent graphs of internal degree k, with cyclic orientation at each trivalent vertex and whose univalent vertices are labelled by elements of H and are totally ordered. These graphs are considered modulo the well-known AS, IHX, multilinearity relations, and up to some "STU-like relations" dealing with the order of the univalent vertices. In the closed case, some relations of a symplectic type can be added. Then, there is a surjective *surgery map*

$$\mathcal{A}_k(H) \xrightarrow{\psi_k} \overline{\mathcal{C}}_k(\Sigma)$$

sending each graph G to $(1_\Sigma)_{\tilde{G}}$, where $\tilde{G}$ is a clover in the manifold 1_Σ with G as associated abstract graph, whose leaves are stacked from the upper surface $\Sigma \times 1$ according to the total order, framed along this surface and embedded according to the labels of the corresponding univalent vertices. The fact that ψ_k is well-defined also follows from the calculus of clovers.

As for the case $k = 1$, Habiro does not define any space of graphs but announces the following isomorphisms

$$\left\{ \begin{array}{l} \overline{\mathcal{C}}_1(\Sigma_{g,1}) \simeq \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2 \\ \overline{\mathcal{C}}_1(\Sigma_g) \simeq \Lambda^3 H/(\omega \wedge H) \oplus \Lambda^2 H_{(2)}/\omega_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2 \end{array} \right. \tag{1.1}$$

where $H_{(2)} = H \otimes \mathbf{Z}_2$ and where

$$\omega = \sum_{i=1}^{g} x_i \wedge y_i \in \Lambda^2 H$$

is the *symplectic element*. This fact has been used afterwards in [**Le**].
The goal of this paper is to prove these isomorphisms, in a diagrammatic way, by again defining a surgery map

$$\mathcal{A}_1(P) \xrightarrow{\psi_1} \overline{\mathcal{C}}_1(\Sigma).$$

[1]This equivalence relation is called $(k-1)$-equivalence in [**Go**], and A_k-equivalence in [**Hr**].

The space of graphs $\mathcal{A}_1(P)$ and the map ψ_1 appear to be meaningfully different from $\mathcal{A}_k(H)$ and ψ_k for $k>1$, making thus the case $k=1$ exceptional. Indeed, their definition will involve both the homology group H and $Spin(\Sigma)$, the set of *spin structures* on Σ.

1.3. The Abelianized Torelli group. We denote by Ω_g the set of *quadratic forms* with $\bullet : H_{(2)} \times H_{(2)} \longrightarrow \mathbf{Z}_2$ as associated bilinear form, namely

$$\Omega_g = \left\{ H_{(2)} \xrightarrow{q} \mathbf{Z}_2 \ : \ \forall x, y \in H_{(2)},\ q(x+y) - q(x) - q(y) = x \bullet y \right\}.$$

Note that Ω_g is an affine space over $H_{(2)}$, with action given by

$$\forall q \in \Omega_g, \forall x \in H_{(2)}, \quad x \cdot q := q + x \bullet (-).$$

Thus, among the maps $\Omega_g \longrightarrow \mathbf{Z}_2$, there are the affine functions, and more generally there are the *Boolean polynomials* which are defined to be sums of products of affine ones (see [**J2**, §4]). These polynomials form a $\mathbf{Z}_2$-algebra denoted by B_g, which is filtered by the *degree* (defined in the obvious way):

$$B_g^{(0)} \subset B_g^{(1)} \subset \cdots \subset B_g.$$

For instance, $B_g^{(1)}$ is the space of affine functions on Ω_g; the constant function $\overline{1} : \Omega_g \longrightarrow \mathbf{Z}_2$ sending each q to 1 and, for $h \in H$, the function $\overline{h}$ sending each q to $q(h)$ are affine functions. Note the following identity:

$$\forall h_1, h_2 \in H, \quad \overline{h_1 + h_2} = \overline{h_1} + \overline{h_2} + (h_1 \bullet h_2) \cdot \overline{1} \in B_g^{(1)}. \tag{1.2}$$

Another example of Boolean polynomial is the quadratic Boolean function

$$\alpha = \sum_{i=1}^{g} \overline{x_i} \cdot \overline{y_i},$$

which is known as the *Arf invariant*. For any basis $(e_i)_{i=1}^{2g}$ for H, there is an isomorphism of algebras:

$$B_g \simeq \frac{\mathbf{Z}_2[t_1, \ldots, t_{2g}]}{t_i^2 = t_i} \tag{1.3}$$

sending $\overline{1}$ to 1 and $\overline{e_i}$ to t_i.

Recall now from [**J2**], that the many *Birman-Craggs homomorphisms* can be summed up into a single homomorphism

$$\mathcal{T}_{g,1} \xrightarrow{\beta} B_g^{(3)} \qquad \text{or} \qquad \mathcal{T}_g \xrightarrow{\beta} \frac{B_g^{(3)}}{\alpha \cdot B_g^{(1)}},$$

according to whether one is considering the boundary case or the closed case. Recall also from [**J3**] that the *first Johnson homomorphism* is a homomorphism

$$\mathcal{T}_{g,1} \xrightarrow{\eta_1} \Lambda^3 H \qquad \text{or} \qquad \mathcal{T}_g \xrightarrow{\eta_1} \frac{\Lambda^3 H}{\omega \wedge H}.$$

Form the following pull-back:

$$\begin{tikzcd}
\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_g^{(3)} \arrow[d, "q"] \\
\Lambda^3 H \arrow[r, "- \otimes \mathbf{Z}_2"'] & \Lambda^3 H_{(2)},
\end{tikzcd}$$

where the map q is the canonical projection $B_g^{(3)} \longrightarrow B_g^{(3)}/B_g^{(2)}$ followed by the isomorphism $B_g^{(3)}/B_g^{(2)} \simeq \Lambda^3 H_{(2)}$ which identifies the cubic polynomial $\overline{h_1 h_2 h_3}$ with

$h_1 \wedge h_2 \wedge h_3$ (this is well-defined because of (1.2) and (1.3)).
We denote by S the subgroup of this pull-back corresponding to $\omega \wedge H \subset \Lambda^3 H$ and $\alpha \cdot B_g^{(1)} \subset B_g^{(3)}$. Johnson has shown in [**J4**] that, under the assumption $g \geq 3$, the homomorphisms η_1 and β induce isomorphisms

$$\frac{\mathcal{T}_{g,1}}{\mathcal{T}'_{g,1}} \xrightarrow[\simeq]{(\eta_1,\beta)} \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} \quad \text{and} \quad \frac{\mathcal{T}_{g}}{\mathcal{T}'_{g}} \xrightarrow[\simeq]{(\eta_1,\beta)} \frac{\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}}{S}.$$

REMARK 1.2. Note that, because of (1.3), the codomains of these maps are respectively non-canonically isomorphic to $\Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$ and $\Lambda^3 H/(\omega \wedge H) \oplus \Lambda^2 H_{(2)}/\omega_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$.

1.4. Statement of the results. In §2, we will construct the space of graphs $\mathcal{A}_1(P)$ and the surgery map $\psi_1 : \mathcal{A}_1(P) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$. Spin structures play a prominent role in their definitions.
Observe that, $\overline{\mathcal{C}}_1(\Sigma)$ being an Abelian group, the mapping cylinder construction induces a group homomorphism

$$\frac{\mathcal{T}(\Sigma)}{\mathcal{T}(\Sigma)'} \xrightarrow{C} \overline{\mathcal{C}}_1(\Sigma).$$

As pointed out by Garoufalidis and Levine in [**GL**] and [**Le**], Johnson homomorphisms and Birman-Craggs homomorphisms factor through $C : \mathcal{T}(\Sigma) \longrightarrow \mathcal{HC}(\Sigma)$. These extensions will be detailed in §3.
Next, we will specify in §4 an isomorphism $\rho : \mathcal{A}_1(P) \longrightarrow \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ and the following two theorems will be proved from the previous material.

THEOREM 1.3. *In the boundary case, the diagram*

$$\begin{tikzcd}
\mathcal{A}_1(P) \arrow[r, "\psi_1"] \arrow[dr, "\rho"'] & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "{(\eta_1,\beta)}"] & \frac{\mathcal{T}_{g,1}}{\mathcal{T}'_{g,1}} \arrow[l, "C"'] \arrow[dl, "{(\eta_1,\beta)}"] \\
& \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} &
\end{tikzcd}$$

commutes and all of its arrows are isomorphisms, *except for the two maps starting from $\mathcal{T}_{g,1}/\mathcal{T}'_{g,1}$ when $g < 3$.*

THEOREM 1.4. *In the closed case, the diagram*

$$\begin{tikzcd}
\frac{\mathcal{A}_1(P)}{\rho^{-1}(S)} \arrow[r, "\psi_1"] \arrow[dr, "\rho"'] & \overline{\mathcal{C}}_1(\Sigma_{g}) \arrow[d, "{(\eta_1,\beta)}"] & \frac{\mathcal{T}_{g}}{\mathcal{T}'_{g}} \arrow[l, "C"'] \arrow[dl, "{(\eta_1,\beta)}"] \\
& \frac{\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}}{S} &
\end{tikzcd}$$

commutes and all of its arrows are isomorphisms, *except for the two maps starting from $\mathcal{T}_g/\mathcal{T}'_g$ when $g < 3$.*

Note that Theorem 1.3 and Theorem 1.4 together with Remark 1.2, give Habiro's isomorphisms (1.1), which are non-canonical. Also, we will easily deduce the following.

COROLLARY 1.5. *For $\Sigma = \Sigma_{g,1}$ or Σ_g, let M and M' be two homology cylinders over Σ. Then, the following assertions are equivalent:*

(a) *M and M' are Y_2-equivalent;*

(b) *M and M' are not distinguished by degree* 1 *Goussarov-Habiro finite type invariants;*
(c) *M and M' are not distinguished by the first Johnson homomorphism nor Birman-Craggs homomorphisms.*

Finally, if an embedding $\Sigma_{g,1} \hookrightarrow \Sigma_g$ is fixed, there is an obvious "filling-up" map $\mathcal{C}_1(\Sigma_{g,1}) \longrightarrow \mathcal{C}_1(\Sigma_g)$, through which the commutative diagrams of Theorem 1.3 and Theorem 1.4 are compatible. The reader is referred to §4 for a precise statement.

2. Definition of the surgery map ψ_1

In this section, we define the space of graphs $\mathcal{A}_1(P)$ and the surgery map ψ_1 announced in the introduction.

2.1. Special Abelian groups and the $\mathcal{A}_1$ functor.

Let us denote by $\mathcal{A}b$ the category of Abelian groups. An *Abelian group with special element* is a pair (G, s) where G is an Abelian group and $s \in G$ is of order at most 2. We denote by $\mathcal{A}b_s$ the category of special Abelian groups whose morphisms are group homomorphisms preserving the special elements. We now define a functor

$$\mathcal{A}b_s \xrightarrow{\mathcal{A}_1} \mathcal{A}b$$

in the following way. For (G, s) an object in $\mathcal{A}b_s$, $\mathcal{A}_1(G, s)$ is the free Abelian group generated by Y-shaped unitrivalent graphs, whose trivalent vertex is equipped with a cyclic order on the incident edges and whose univalent vertices are labelled by G, subject to some relations. The notation

$$\mathsf{Y}[z_1, z_2, z_3]$$

will stand for the Y-shaped graph whose univalent vertices are colored by z_1, z_2 and $z_3 \in G$ in accordance with the cyclic order, so that our notation is invariant under cyclic permutation of the z_i's. The relations are the following ones:

$$\begin{aligned} \textbf{Antisymmetry (AS)} \quad &: \quad \mathsf{Y}[z_1, z_2, z_3] = -\mathsf{Y}[z_2, z_1, z_3], \\ \textbf{Multilinearity of colors} \quad &: \quad \mathsf{Y}[z_0 + z_1, z_2, z_3] = \mathsf{Y}[z_0, z_2, z_3] + \mathsf{Y}[z_1, z_2, z_3], \\ \textbf{Slide} \quad &: \quad \mathsf{Y}[z_1, z_1, z_2] = \mathsf{Y}[s, z_1, z_2], \end{aligned}$$

where $z_0, z_1, z_2, z_3 \in G$. For $(G, s) \xrightarrow{f} (G', s')$ a morphism in $\mathcal{A}b_s$, $\mathcal{A}_1(f)$ maps each generator $\mathsf{Y}[z_1, z_2, z_3]$ of $\mathcal{A}_1(G, s)$ to $\mathsf{Y}[f(z_1), f(z_2), f(z_3)] \in \mathcal{A}_1(G', s')$.

EXAMPLE 2.1. The map $[G \longmapsto (G, 0)]$ makes $\mathcal{A}b$ a (full) subcategory of $\mathcal{A}b_s$. It follows from the definitions that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}b \arrow[r, tail] \arrow[dr, "\Lambda^3(-)"'] & \mathcal{A}b_s \arrow[d, "\mathcal{A}_1"] \\ & \mathcal{A}b. \end{tikzcd}$$

Non-trivial examples will be given in the next paragraph. For future use, note that this category has an obvious pull-back construction extending that of $\mathcal{A}b$:

$$\begin{tikzcd}
(G_1,s_1)\times_{(G,s)}(G_2,s_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (G_2,s_2) \arrow[d, "f_2"] \\
(G_1,s_1) \arrow[r, "f_1"'] & (G,s)
\end{tikzcd}$$

where $(G_1,s_1)\times_{(G,s)}(G_2,s_2)$ is the subgroup of $G_1\times G_2$ consisting of those (z_1,z_2) such that $f_1(z_1)=f_2(z_2)$, and with special element (s_1,s_2).

2.2. Spin structures and the special Abelian group P. In this paragraph, let M be a compact oriented 3-manifold endowed with a Riemannian metric, and let FM be its bundle of oriented orthonormal frames:

$$SO(3) \overset{i}{\rightarrowtail} E(FM) \overset{p}{\twoheadrightarrow} M.$$

Let $s\in H_1(E(FM);\mathbf{Z})$ be the image by i_* of the generator of $H_1(SO(3);\mathbf{Z})\simeq\mathbf{Z}_2$. Recall that M is spinnable and that $Spin(M)$ can be defined as

$$Spin(M)=\left\{y\in H^1(E(FM);\mathbf{Z}_2),\ <y,s>\neq 0\right\},$$

which is essentially independent of the metric. The manifold M being spinnable, s is not 0 (and so is of order 2).

Now, $Spin(M)$ being an affine space over $H^1(M;\mathbf{Z}_2)$ with action given by

$$\forall x\in H^1(M;\mathbf{Z}_2),\forall\sigma\in Spin(M),\quad x\cdot\sigma:=\sigma+p^*(x),$$

we can consider the space

$$A\left(Spin(M),\mathbf{Z}_2\right)$$

of $\mathbf{Z}_2$-valued affine functions on $Spin(M)$. For instance, $\overline{1}\in A\left(Spin(M),\mathbf{Z}_2\right)$ will denote the constant map defined by $\sigma\longmapsto 1$.
There is a canonical map

$$A\left(Spin(M),\mathbf{Z}_2\right)\xrightarrow{\kappa}H_1(M;\mathbf{Z}_2).$$

For $f\in A\left(Spin(M),\mathbf{Z}_2\right)$, the homology class $\kappa(f)$ is defined unambiguously by

$$\forall\sigma,\sigma'\in Spin(M),\quad f(\sigma')-f(\sigma)=<\sigma'/\sigma,\kappa(f)>\in\mathbf{Z}_2,$$

where $\sigma'/\sigma\in H^1(M;\mathbf{Z}_2)$ is defined by the affine action of $H^1(M;\mathbf{Z}_2)$ on $Spin(M)$. Another canonical map is

$$H_1(E(FM);\mathbf{Z})\xrightarrow{e}A\left(Spin(M),\mathbf{Z}_2\right)$$

sending a x to the map defined by $\sigma\longmapsto\ <\sigma,x>$. Next lemma gives us a nice understanding of the special Abelian group $\left(H_1(E(FM);\mathbf{Z}),s\right)$.

LEMMA 2.2. *a) The following diagram of special groups is a pull-back diagram:*

$$\begin{tikzcd}
\left(H_1(E(FM);\mathbf{Z}),s\right) \arrow[r, "e"] \arrow[d, "p_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \left(A\left(Spin(M),\mathbf{Z}_2\right),\overline{1}\right) \arrow[d, "\kappa"] \\
\left(H_1(M;\mathbf{Z}),0\right) \arrow[r, "-\otimes\mathbf{Z}_2"'] & \left(H_1(M;\mathbf{Z}_2),0\right).
\end{tikzcd}$$

b) Let t be the map

$$\left\{\textit{Oriented framed knots in } M\right\}\xrightarrow{t}H_1(E(FM);\mathbf{Z})$$

which adds to any oriented framed knot K an extra $(+1)$-twist, and next sends it to the homology class of its lift in FM. Then,

(i) *t is surjective;*
(ii) *$t_{K_1} = t_{K_2}$ if and only if K_1 and K_2 are cobordant as oriented knots in M and if their framings with respect to a surface with boundary $(K_1) \cup (-K_2)$ then differ from each other by an even integer;*
(iii) *if $K_1 \sharp K_2$ denotes the band connected sum of K_1 and K_2, then $t_{K_1 \sharp K_2} = t_{K_1} + t_{K_2}$;*
(iv) *the k-framed trivial oriented knot ($k \in \mathbf{Z}$) is sent by t to $k \cdot s$.*

PROOF. We begin by proving a). The commutativity of the diagram of special groups is easy to verify. By functoriality, we get a map

$$\left(H_1\left(E(FM);\mathbf{Z}\right), s\right) \xrightarrow{(p_*, e)} \left(H_1\left(M;\mathbf{Z}\right), 0\right) \times_{\left(H_1(M;\mathbf{Z}_2), 0\right)} \left(A\left(Spin(M), \mathbf{Z}_2\right), \overline{1}\right).$$

The Serre sequence associated to the fibration FM gives for homology with integer coefficients:

$$0 \longrightarrow H_1(SO(3);\mathbf{Z}) \xrightarrow{i_*} H_1(E(FM);\mathbf{Z}) \xrightarrow{p_*} H_1(M;\mathbf{Z}) \longrightarrow 0.$$

The bijectivity of (p_*, e) follows from the exactness of this sequence.

We now prove b) and we begin with assertion (iv). Let K be a trivial k-framed oriented knot, let $* \in K$ and let $e = (e_1, e_2, e_3) \in p^{-1}(*)$ be the framing of K at $*$. We denote by $\tilde{K}$ the lift of K to FM. Then, as a loop in $E(FM)$, $\tilde{K}$ is homotopic to the loop in the fiber $p^{-1}(*)$ defined by

$$[0,1] \ni t \longmapsto R_{2\pi(k+1)t}(e),$$

where R_θ (with $\theta \in \mathbf{R}$) denotes the rotation of oriented axis directed by e_3 and angle θ. From an appropriate description of the generator of $\pi_1\left(SO(3)\right) \simeq \mathbf{Z}_2$, it follows that $\left[\tilde{K}\right] = (k+1) \cdot s \in H_1(E(FM);\mathbf{Z})$, and assertion (iv) then follows.

Let us make an observation. Let K be any oriented framed knot in M; since the framing of K determines a trivialization of its normal bundle in M, it allows us to restrict any spin structure on M to K. Recall now that the cobordism group Ω_1^{Spin} is isomorphic to $\mathbf{Z}_2$ (with generator given by $\mathbf{S}^1$ endowed with the spin structure induced by its Lie group structure: see [**Ki**, p. 35, 36]). The following observation then makes sense:

$$\forall \sigma \in Spin(M), \quad e(t_K)(\sigma) = (K, \sigma|_K) \ \in \Omega_1^{Spin} \simeq \mathbf{Z}_2, \tag{2.1}$$

and can be derived from an appropriate characterization of the spin structures on the circle (see [**Ki**, p. 35, 36]).

Let now K_1 and K_2 be some disjoint oriented framed knots in M. There is an obvious genus 0 surface with boundary $K_1 \sharp K_2 \dot{\cup} (-K_1) \dot{\cup} (-K_2)$. Then, according to (2.1), we have $e(t_{K_1 \sharp K_2}) = e(t_{K_1}) + e(t_{K_2})$. Also, $p_*\left(t_{K_1 \sharp K_2}\right) = [K_1 \sharp K_2] = [K_1] + [K_2] = p_*\left(t_{K_1}\right) + p_*\left(t_{K_2}\right)$, and so by a), we obtain that assertion (iii) holds for K_1 and K_2.

We now justify assertion (ii). According to a), $t_{K_1} = t_{K_2}$ if and only if $p_*(t_{K_1}) = p_*(t_{K_2})$ and $e(t_{K_1}) = e(t_{K_2})$. Also, the condition $p_*(t_{K_1}) = p_*(t_{K_2})$ holds if and only if K_1 and K_2 are homologous in M. In this case, let S be an embedded oriented surface in M such that $\partial S = K_1 \dot{\cup} (-K_2)$. Let k_i be the framing of K_i with respect to S and let K_i' be the oriented framed knot obtained from K_i by adding an extra $(-k_i)$-twist, so that the framing of K_i' is given by S. Then, according to (2.1), we have $e\left(t_{K_1'}\right) = e\left(t_{K_2'}\right)$. Moreover, applying assertions (iii) and (iv), we obtain: $e\left(t_{K_i'}\right) = e\left(t_{K_i}\right) + k_i \cdot s$. We conclude that $e\left(t_{K_1}\right) = e\left(t_{K_2}\right)$ if and only if k_1 and k_2 are equal modulo 2, proving thus assertion (ii).

Let $x \in H_1(E(FM);\mathbf{Z})$, then $p_*(x) \in H_1(M;\mathbf{Z})$ can be realized by an oriented knot K in M: we give it an arbitrary framing. By construction, $p_*(t_K - x) = 0 \in H_1(M;\mathbf{Z})$, and so by exactness of the Serre sequence, $t_K - x = \varepsilon \cdot s$ with $\varepsilon \in \{0,1\}$. By possibly band-summing K with a trivial $(+1)$-framed knot when $\varepsilon = 1$, and according to assertion (iii) and (iv), the framed knot K can be supposed to be such that $t_K = x$; this proves assertion (i). $\square$

We now restrict ourselves to the 3-manifold $M = 1_\Sigma = \Sigma \times I$ where Σ can be Σ_g or $\Sigma_{g,1}$. The inclusion $i^+ : \Sigma \hookrightarrow 1_\Sigma$, with image Σ^+, induces an isomorphism between H and $H_1(M;\mathbf{Z})$ and a bijection between $Spin(\Sigma)$ and $Spin(M)$. As shown by Johnson in [**J1**], there is an algebraic way to think of $Spin(\Sigma)$. Indeed, there exists a canonical affine isomorphism

$$Spin(\Sigma) \xrightarrow{\simeq} \Omega_g,$$

sending any spin structure σ to a quadratic form q_σ which can be defined as follows. Let $x \in H_{(2)} = H_1(\Sigma;\mathbf{Z}_2)$ be represented by an oriented simple closed curve on Σ^+; by framing it along Σ^+ and pushing it into the interior of 1_Σ, we get a framed oriented knot K in 1_Σ. Then,

$$q_\sigma(x) = e(t_K)\,(\sigma \times I) \ \in \mathbf{Z}_2. \tag{2.2}$$

Therefore, according to Lemma 2.2 a), $(H_1\left(E\left(F1_\Sigma\right);\mathbf{Z}\right), s)$ is canonically isomorphic to the special Abelian group defined by the pull-back construction

$$\begin{tikzcd}
(H,0) \times_{(H_{(2)},0)} \left(B_g^{(1)}, \overline{1}\right) \arrow[r, "e"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \left(B_g^{(1)}, \overline{1}\right) \arrow[d, "\kappa"] \\
(H,0) \arrow[r, "- \otimes \mathbf{Z}_2"'] & (H_{(2)},0)
\end{tikzcd}$$

whose projections are denoted by p and e, and where κ is the composite

$$B_g^{(1)} \twoheadrightarrow B_g^{(1)}/B_g^{(0)} \xrightarrow{\simeq} H_{(2)}.$$

The last isomorphism here identifies $\overline{h}$ with $h_{(2)}$ for all $h \in H$ (this is well-defined by (1.2) and (1.3)). We define the special Abelian group P to be

$$\boxed{P = (H,0) \times_{(H_{(2)},0)} \left(B_g^{(1)}, \overline{1}\right),}$$

and $\mathcal{A}_1(P)$ is the space of graphs announced in the introduction.

Remark 2.3. Thus, any element z of P can be written as

$$z = \left(h, \overline{h} + \varepsilon \cdot \overline{1}\right) \in P,$$

with $h \in H$ and $\varepsilon \in \{0,1\}$. Observe also the following. Suppose that there exists a simple oriented closed curve in Σ^+ with homology class h. Let K be the push-off of this curve, framed along Σ^+, with an extra ε-twist. Then, it follows from (2.2) that $t_K = z \in P \simeq H_1\left(E\left(F1_\Sigma\right);\mathbf{Z}\right)$.

Remark 2.4. According to the proof of Lemma 2.2, the Serre sequence for homology associated to the bundle $F1_\Sigma$ gives the following short exact sequence:

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow P \xrightarrow{p} H \longrightarrow 0,$$

where $\mathbf{Z}_2$ is injected into P by sending 1 to $(0,\overline{1})$. The map $s : H \longrightarrow P$ defined by $s(h) = \left(h, \overline{h}\right)$ is a set-theoretic section. According to (1.2), the associated 2-cocycle

$H \times H \longrightarrow \mathbf{Z}_2$ is the mod 2 reduced intersection form of Σ. Thus, P is isomorphic to $H \rtimes \mathbf{Z}_2$ with crossed product defined by

$$(h_1, \varepsilon_1) \cdot (h_2, \varepsilon_2) = (h_1 + h_2, \varepsilon_1 + \varepsilon_2 + h_1 \bullet h_2).$$

The element $\left(h, \overline{h} + \varepsilon \cdot \overline{1}\right) \in P$ corresponds to $(h, \epsilon) \in H \rtimes \mathbf{Z}_2$.

2.3. The surgery map ψ_1. In this paragraph, Σ is allowed to be Σ_g or $\Sigma_{g,1}$ and the surgery map $\psi_1 : \mathcal{A}_1(P) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$ is constructed by means of calculi of clovers.

CONVENTIONS 2.5. Here, we *adopt* Goussarov's convention for the surgery meaning of Y-graphs and clovers [**Go**], [**GGP**].

Denote by $\tilde{\mathcal{A}}_1(P)$ the free Abelian group generated by abstract Y-shaped graphs whose univalent vertices are labelled by P, and which are equipped with an orientation at their trivalent vertex: $\mathcal{A}_1(P)$ is a quotient of $\tilde{\mathcal{A}}_1(P)$. For each generator $\mathsf{Y}[z_1, z_2, z_3]$ of $\tilde{\mathcal{A}}_1(P)$, where $z_i \in P$, pick some disjoint oriented framed knots K_i in the interior of 1_Σ such that $t_{K_i} = z_i \in P \simeq H_1\left(E\left(F1_\Sigma\right); \mathbf{Z}\right)$; this is possible according to Lemma 2.2 b) (i). Next, pick an embedded 2-disk D in the interior of 1_Σ and disjoint from the K_i's, orient it in an arbitrary way, and connect it to the K_i's with some bands e_i. These band sums are required to be compatible with the orientations, and to be coherent with the cyclic ordering $(1, 2, 3)$. See Fig. 2.1 as an illustration. What we obtain in 1_Σ is precisely a *Y-graph*, as defined by

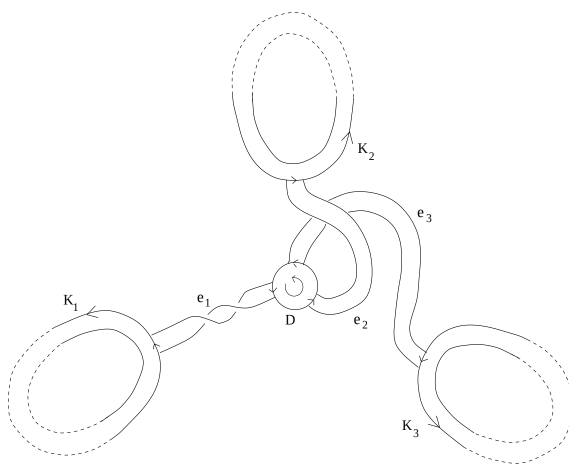

Figure 2.1: Embedding the Y-graph

Goussarov in [**Go**]. We denote it by $\phi\big(\mathsf{Y}[z_1, z_2, z_3]\big)$. For example, as follows from Lemma 2.2 b) (iv), if z_1 is the special element s of P, the corresponding leaf K_1 of $\phi\big(\mathsf{Y}[s, z_2, z_3]\big)$ can be chosen to be unknotted and $(+1)$-framed; such a leaf is called *special* in [**GGP**].

We now put $\tilde{\psi}_1\big(\mathsf{Y}[z_1, z_2, z_3]\big)$ to be the Y_2-equivalence class of the surgered manifold $(1_\Sigma)_{\phi(\mathsf{Y}[z_1, z_2, z_3])}$, so that we get a map

$$\tilde{\mathcal{A}}_1(P) \xrightarrow{\tilde{\psi}_1} \overline{\mathcal{C}}_1(\Sigma).$$

THEOREM 2.6. *The map $\tilde{\psi}_1$ does not depend on the choice of ϕ, and induces a surjective quotient map*

$$\mathcal{A}_1(P) \overset{\psi_1}{\twoheadrightarrow} \overline{\mathcal{C}}_1(\Sigma).$$

PROOF. The proof might be read with a copy of [**GGP**] in hand. Using the above notation, we begin with showing that $\tilde{\psi}_1(\mathsf{Y}[z_1, z_2, z_3])$ does not depend on the choice of $\phi(\mathsf{Y}[z_1, z_2, z_3])$. For this, we recall two facts concerning any Y-graph G in a homology cylinder M (see Remark 2.7 below):

Fact 1: the Y_2-equivalence class of M_G is not modified when an edge of G is band-summed with a (disjoint) oriented framed knot of M;

Fact 2: the Y_2-equivalence class of M_G is inverted when an edge of G is half-twisted.

Using these, the independance of the choice of the disk D, its orientation and the edges e_i is easily shown.
We now show the independance of the choice of the leaves K_i. Suppose for example that K_1' is another choice of K_1. Then, according to Lemma 2.2 b) (ii), there exists an embedded oriented surface F in 1_Σ such that $\partial F = K_1 \dot{\cup} (-K_1')$ and such that, if k (resp. k') is the framing of K_1 (resp. K_1') with respect to F, $(k - k')$ is even. We also assume transversality of F with the edges of the Y-graph, and with the two other leaves K_2 and K_3. Let $g(F)$ denote the genus of F, let m be the number of intersection points of F with the edges, and for $i = 2, 3$, let n_i be the number of intersection points of F with K_i. If all of the integers $g(F)$, $(k - k')$, m, n_2 and n_3 are zero, the two Y-graphs are isotopic and we are done. In the general case, recall from [**GGP**, §4.3] that there is a procedure for *simplifying the leaves*. The main tool for this is the following:

Fact 3: if G_1 and G_2 are two Y-graphs in 1_Σ obtained from a Y-graph G by *splitting a leaf*, then $(1_\Sigma)_G = (1_\Sigma)_{G_1} \cdot (1_\Sigma)_{G_2} \in \overline{\mathcal{C}}_1(\Sigma)$ (see Remark 2.7).

Splitting $\big(g(F) + |k - k'|/2 + m + n_2 + n_3\big)$ times the leaf K_1, splitting n_2 times the leaf K_2 and splitting n_3 times the leaf K_3, we see that the result $\tilde{\psi}_1(\mathsf{Y}[z_1, z_2, z_3])$ in $\overline{\mathcal{C}}_1(\Sigma)$ defined by the choice of K_1 differs from the one defined by K_1' by some elements of the form $(1_\Sigma)_G$, where G satisfies one of the following conditions:

(i) G has a leaf which bounds a genus 1 surface disjoint from G and with respect to which the leaf is 0-framed;
(ii) G has a leaf which bounds a disk disjoint from G, and with respect to which the leaf is (± 2)-framed;
(iii) G has a leaf which bounds a disk with respect to which it is 0-framed, and this disk intersects G in exactly one point belonging to an edge;
(iv) G has two leaves which are linked as the Hopf link.

Let us now verify that all of these elements vanish in $\overline{\mathcal{C}}_1(\Sigma)$. If G is of type (i), the surgery effect of G is the same as a clover of degree 2 (apply [**GGP**, Lem. 5.1] and [**GGP**, Th. 2.4]). If G is of type (ii), by again cutting its leaf we get $(1_\Sigma)_G = 2 \cdot (1_\Sigma)_{G'}$ where G' has a special leaf; but $(1_\Sigma)_{G'} = -(1_\Sigma)_{G'}$ by Fact 2. If G is of type (iii), by applying Fact 1 the edge can be slid away from the leaf, we then get a Y-graph with a *trivial* leaf which has no surgery effect by the "blow-up move" of [**GGP**, Fig. 6]. If G is of type (iv), by applying [**GGP**, Th. 2.4], we obtain a Y-graph with a looped edge, but this is stated to be 0 in $\overline{\mathcal{C}}_1(\Sigma)$ by the so-called *LOOP relation*. This relation is easily shown from [**GGP**, Lem. 2.3] and from Fact 1 and Fact 2. This completes the proof of the independance of $\tilde{\psi}_1$ of ϕ.

The fact that $\tilde{\psi}_1$ is surjective follows immediately from the fact that the Abelian group $\overline{\mathcal{C}}_1(\Sigma)$ is generated by the homology cylinders $(1_\Sigma)_G$ where G is a single Y-graph (this is also proved by standard calculi of clovers).

We now show that the map $\tilde{\psi}_1$ factors through $\tilde{\mathcal{A}}_1(P) \longrightarrow \mathcal{A}_1(P)$. The AS relation is proved in $\overline{\mathcal{C}}_1(\Sigma)$ from Fact 2 and an isotopy of the Y-graph – see [**GGP**, Cor. 4.6].
The multilinearity relation follows from Fact 3. Indeed, let G be a Y-graph in

1_Σ with K as a leaf. Split the leaf K to K_1 and K_2, and let G_1 and G_2 be the corresponding new Y-graphs. Then, $(1_\Sigma)_G = (1_\Sigma)_{G_1} \cdot (1_\Sigma)_{G_2} \in \overline{\mathcal{C}}_1(\Sigma)$. Since K is the band connected sum of K_1 and K_2, we have by Lemma 2.2 b) (iii): $t_K = t_{K_1} + t_{K_2} \in P$.
The slide relation is shown to be satisfied in $\overline{\mathcal{C}}_1(\Sigma)$ thanks to the "leaf slide" move of [**GGP**, Fig. 6]. For this, let G be a Y-graph in 1_Σ with some leaves K_1 and K_2 such that $t_{K_1} = -t_{K_2}$. By sliding the leaf K_2 along K_1, we obtain a new Y-graph G' with the same surgery effect as G, such that $K_1' = K_1$ and such that K_2' is the band connected sum of K_1 and K_2 with an extra (-1)-twist. So, by Lemma 2.2 b) (iii) and (iv), we have $t_{K_2'} = t_{K_1} + t_{K_2} + s = s \in P$. This shows that the relation $\mathsf{Y}[z_1, -z_1, z_3] = \mathsf{Y}[z_1, s, z_3]$ $(z_1, z_3 \in P)$ is satisfied in $\overline{\mathcal{C}}_1(\Sigma)$. The slide relation, as stated in §2.1, follows then from the AS and multilinearity relations. $\square$

REMARK 2.7. The proof of Fact 1, Fact 2 and Fact 3 use calculus of clovers and can respectively be obtained from the proof of Cor. 4.2, Lem. 4.4 and Cor. 4.3 in [**GGP**]. Alternatively, those facts can be considered as corollaries of these results in the following way. Denote by $\mathbf{Z}\mathcal{C}_1(\Sigma)$ the free Abelian group generated by the set $\mathcal{C}_1(\Sigma)$, and let

$$\mathbf{Z}\mathcal{C}_1(\Sigma) = \mathcal{F}_0^Y(1_\Sigma) \supset \mathcal{F}_1^Y(1_\Sigma) \supset \mathcal{F}_2^Y(1_\Sigma) \supset \cdots$$

be its Goussarov-Habiro filtration [**GGP**, §1.4]. Results are stated in [**GGP**] to hold in the graded space $\mathcal{G}_k(1_\Sigma) = \mathcal{F}_k^Y(1_\Sigma)/\mathcal{F}_{k+1}^Y(1_\Sigma)$. Consider also the homomorphism of Abelian groups

$$\mathbf{Z}\mathcal{C}_1(\Sigma) \xrightarrow{v} \overline{\mathcal{C}}_1(\Sigma)$$

which assigns to any homology cylinder its Y_2-equivalence class. The invariant v is primitive, in the sense that it restricts to $\mathcal{C}_1(\Sigma)$ to a monoid homomorphism, and is a degree 1 invariant[2], as follows from calculus of clovers. In particular, v induces a homomorphism $\mathcal{G}_1(1_\Sigma) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$, by which Fact 1, Fact 2 and Fact 3 are respectively the images of Cor. 4.2, Lem. 4.4 and Cor. 4.3.

3. Johnson homomorphism and Birman-Craggs homomorphisms for homology cylinders

In this section, the first Johnson homomorphism and the Birman-Craggs homomorphisms are extended to the monoid of homology cylinders.

3.1. The first Johnson homomorphism for homology cylinders. In [**GL**] the notion of Johnson homomorphisms for homology cobordisms over $\Sigma_{g,1}$ was introduced. In this paragraph, we allow Σ to be Σ_g or $\Sigma_{g,1}$, and give the definition of the first Johnson homomorphism in both cases.

The fundamental group of Σ with base point $* \in \Sigma$ will be denoted by $\pi^{(*)}$, and $\pi_k^{(*)}$ will denote the k^{th} term of its lower central series, beginning at $\pi_1^{(*)} = \pi^{(*)}$. We denote by $(x_i, y_i)_{i=1}^g$ the based loops depicted in Fig. 3.1 or their corresponding images under an inclusion $\Sigma_{g,1} \subset \Sigma_g$. Then,

in the boundary case, $\quad \pi^{(*)} = F(x_1, \ldots, x_g, y_1, \ldots, y_g)$,

and in the closed case, $\quad \pi^{(*)} = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \mid \prod_{i=1}^g [x_i, y_i] = 1 \rangle$.

Given a homology cobordism $(M, i^+, i^-) \in \mathcal{C}(\Sigma)$, the map $i^\pm$ induces an isomorphism at the level of each nilpotent quotient (by Stallings [**St**]). We choose a

[2] In fact, v is a universal degree 1 primitive invariant for homology cylinders. See [**Hr**, §6.4] for a similar invariant, of any degree, for knots in the 3-sphere.

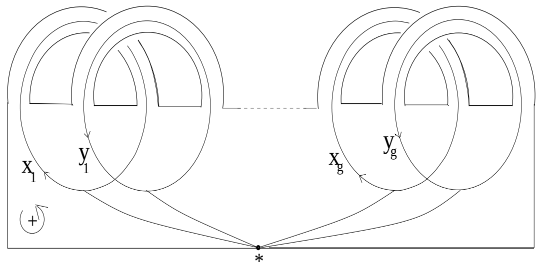

Figure 3.1: The based curves $(x_i, y_i)_{i=1}^{g}$ on $\Sigma_{g,1}$

path $\gamma \subset M$ going from $i^+(*)$ to $i^-(*)$, and then consider the following composite:

$$\frac{\pi^{(*)}}{\pi_3^{(*)}} \xrightarrow[\simeq]{i_3^+} \frac{\pi_1(M, i^+(*))}{\pi_1(M, i^+(*))_3} \xrightarrow[\simeq]{c_\gamma} \frac{\pi_1(M, i^-(*))}{\pi_1(M, i^-(*))_3} \xrightarrow[\simeq]{(i_3^-)^{-1}} \frac{\pi^{(*)}}{\pi_3^{(*)}}.$$

Up to inner automorphisms, this is independent on the choice of γ, so that there is a well-defined map

$$\mathcal{C}(\Sigma) \xrightarrow{\eta_1^{(*)}} Out\left(\frac{\pi^{(*)}}{\pi_3^{(*)}}\right),$$

satisfying $\eta_1^{(*)}(M \cdot N) = \eta_1^{(*)}(N) \cdot \eta_1^{(*)}(M)$. Let $\star$ be another base point in Σ, and γ an arbitrary path between $*$ and $\star$. Conjugation by γ induces an isomorphism $Out\left(\pi^{(*)}/\pi_3^{(*)}\right) \simeq Out\left(\pi^{(\star)}/\pi_3^{(\star)}\right)$. This isomorphism is independent on the choice of the path γ, and the maps $\eta_1^{(*)}$ and $\eta_1^{(\star)}$ are compatible through it. Therefore, we get a well-defined group denoted by $Out(\pi/\pi_3)$ and an anti-homomorphism of monoids

$$\mathcal{C}(\Sigma) \xrightarrow{\eta_1} Out\left(\frac{\pi}{\pi_3}\right). \tag{3.1}$$

If we restrict ourselves to homology cylinders, we are led to a map

$$\mathcal{C}_1(\Sigma) \xrightarrow{\eta_1} Ker\left(Out\left(\frac{\pi}{\pi_3}\right) \to Out\left(\frac{\pi}{\pi_2}\right)\right).$$

Observe the following exact sequence:

$$1 \longrightarrow Hom\left(H, \frac{\pi_2^{(*)}}{\pi_3^{(*)}}\right) \longrightarrow Aut\left(\frac{\pi^{(*)}}{\pi_3^{(*)}}\right) \longrightarrow Aut\left(\frac{\pi^{(*)}}{\pi_2^{(*)}}\right)$$

where any $f \in Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)$ is sent to the automorphism of $\pi^{(*)}/\pi_3^{(*)}$ which sends $\overline{x}$ to $\overline{x}f(\overline{x})$ (with $x \in \pi^{(*)}$). Hence we have the following exact sequence:

$$1 \longrightarrow \frac{Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)}{[H, -]} \longrightarrow Out\left(\frac{\pi}{\pi_3}\right) \longrightarrow Out\left(\frac{\pi}{\pi_2}\right).$$

Here, $[H, -]$ stands for the subgroup of $Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)$ consisting of those homomorphisms $[h, -]$ defined for any $h \in H$ by $x \longmapsto [h, x]$, where H is identified with $\pi_1^{(*)}/\pi_2^{(*)}$. Consequently, we have defined an anti-homomorphism of monoids

$$\mathcal{C}_1(\Sigma) \xrightarrow{\eta_1} \frac{Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)}{[H, -]}.$$

In the sequel, we denote by $\mathsf{L}(H) = \oplus_n \mathsf{L}_n(H)$, the free Lie $\mathbf{Z}$-algebra on the $\mathbf{Z}$-module H, and distinguish the boundary case from the closed case.

In the boundary case, as $\pi^{(*)}$ is free and H is the Abelianized of $\pi^{(*)}$, $\mathsf{L}_2(H)$ is canonically isomorphic to $\pi_2^{(*)}/\pi_3^{(*)}$. Also, there is a sequence of isomorphisms $Hom\,(H, \mathsf{L}_2(H)) \simeq H^* \otimes \mathsf{L}_2(H) \simeq H \otimes \mathsf{L}_2(H)$, with last one induced by $\bullet$-duality. Through these, $[H,-] \subset Hom\,(H, \mathsf{L}_2(H))$ becomes $A_{g,1} \subset H \otimes \mathsf{L}_2(H)$ defined by

$$A_{g,1} = \left\{ \sum_{i=1}^{g} (x_i \otimes [h, y_i] - y_i \otimes [h, x_i]) \mid h \in H \right\}.$$

Thus, η_1 takes values in

$$\frac{Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)}{[H,-]} \simeq \frac{H \otimes \mathsf{L}_2(H)}{A_{g,1}}.$$

The group $\Lambda^3 H$ can be seen as a subgroup of $H \otimes \mathsf{L}_2(H)$ in the following manner:

$$0 \longrightarrow \Lambda^3 H \xrightarrow{\nu} H \otimes \mathsf{L}_2(H) \xrightarrow{[-,-]} \mathsf{L}_3(H),$$

where ν is defined by $\nu(x \wedge y \wedge z) = x \otimes [y,z] + y \otimes [z,x] + z \otimes [x,y]$. Composing ν with the projection $H \otimes \mathsf{L}_2(H) \twoheadrightarrow H \otimes \mathsf{L}_2(H)/A_{g,1}$ still gives an injection

$$\Lambda^3 H \overset{\nu}{\rightarrowtail} \frac{H \otimes \mathsf{L}_2(H)}{A_{g,1}}.$$

This follows from the fact that

$$\forall h \in H, \quad [h, \omega] = 0 \in \mathsf{L}_3(H) \implies h = 0, \tag{3.2}$$

where $\omega = \sum_i [x_i, y_i] \in \mathsf{L}_2(H)$ corresponds *via* the canonical isomorphism $\mathsf{L}_2(H) \simeq \Lambda^2 H$ to the symplectic element ω, defined in the introduction.
We now prove that η_1 takes values in the subgroup $\Lambda^3 H$. Suppose for this that $f \in Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right) \subset Aut\left(\pi^{(*)}/\pi_3^{(*)}\right)$ is such that there exists a lift $\tilde{f} \in End(\pi^{(*)})$ of f fixing the boundary element $\partial := \prod_{i=1}^{g} [x_i, y_i]$ modulo $\pi_4^{(*)}$. Note that this property is verified by a representative for $\eta_1(M)$ if M is a homology cylinder, so that proving that $f \in Ker([-,-])$ will prove that $Im(\eta_1) \subset \Lambda^3 H$. Let $X_i = x_i^{-1}\tilde{f}(x_i) \in \pi_2^{(*)}$ and $Y_i = y_i^{-1}\tilde{f}(y_i) \in \pi_2^{(*)}$. We have

$$\begin{array}{rcl} \tilde{f}(\partial) & = & \prod_i [\tilde{f}(x_i), \tilde{f}(y_i)] \\ & \equiv & \prod_i [x_i X_i, y_i Y_i] \\ & \equiv & \prod_i [x_i, y_i][X_i, y_i][x_i, Y_i] \quad \mod \pi_4^{(*)}, \end{array}$$

which implies that $\prod_i [X_i, y_i][x_i, Y_i] \equiv 1 \mod \pi_4^{(*)}$. Consequently,

$$\sum_i (x_i \otimes Y_i - y_i \otimes X_i) \in H \otimes \mathsf{L}_2(H),$$

which essentially corresponds to f, goes to 0 by the bracketting map.

Let us now focus on the closed case. The canonical map $\mathsf{L}_2(H) \twoheadrightarrow \pi_2^{(*)}/\pi_3^{(*)}$ induces an isomorphism between $\pi_2^{(*)}/\pi_3^{(*)}$ and $\mathsf{L}_2(H)/\omega$. Thus, in this case, η_1 takes values in

$$\frac{Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)}{[H,-]} \simeq \frac{H \otimes \mathsf{L}_2(H)}{A_g}$$

where $A_g = A_{g,1} + H \otimes \omega$. Since $\nu\,(\omega \wedge H) \subset A_g$, ν factors to give

$$\frac{\Lambda^3 H}{\omega \wedge H} \xrightarrow{\ \nu\ } \frac{H \otimes \mathsf{L}_2(H)}{A_g}.$$

It also follows from (3.2) that this new ν is still injective. Then, $\Lambda^3 H/\omega \wedge H$ can be seen as a subgroup of $Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)/[H,-]$. Similarly to the boundary case, one shows that η_1 takes values in $\Lambda^3 H/\omega \wedge H$.

So far, we have defined some anti-homomorphisms of monoids

$$\mathcal{C}_1(\Sigma_{g,1}) \xrightarrow{\ \eta_1\ } \Lambda^3 H \quad \text{and} \quad \mathcal{C}_1(\Sigma_g) \xrightarrow{\ \eta_1\ } \frac{\Lambda^3 H}{\omega \wedge H},$$

but next lemma allows us to go a bit further.

LEMMA 3.1. *Let (M,K) be a homology cylinder over Σ together with a loop K based on $* \in M$. Let also G be a degree 2 clover in M disjoint from K and let (M_G, K_G) be the result of the surgery along G. Then, there exists an isomorphism*

$$\frac{\pi_1(M,*)}{\pi_1(M,*)_3} \xrightarrow{\ \simeq\ } \frac{\pi_1(M_G,*)}{\pi_1(M_G,*)_3}$$

sending $[K]$ to $[K_G]$.

This lemma allows us to conclude with the following proposition-definition.

PROPOSITION 3.2. *For homology cylinders over $\Sigma = \Sigma_{g,1}$ or Σ_g, there are some well-defined homomorphisms*

$$\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{\ \eta_1\ } \Lambda^3 H \quad \textit{and} \quad \overline{\mathcal{C}}_1(\Sigma_g) \xrightarrow{\ \eta_1\ } \frac{\Lambda^3 H}{\omega \wedge H}.$$

Induced by the map (3.1), they are called the first Johnson homomorphisms*.*

REMARK 3.3. The composition of η_1 with the map $C : \mathcal{T}(\Sigma) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$ is the classical homomorphism defined in [**J3**].

PROOF OF LEMMA 3.1. Using [**GGP**, Lem. 5.1], one shows that

$$M_G \cong_+ M \setminus int(N(G)) \ \cup_{j|_\partial} \ (H_4)_L$$

where $H_4 \overset{j}{\hookrightarrow} M$ is an oriented embedding of the standard genus 4 handlebody onto $N(G)$, which is a regular neighborhood of G in M, and where L is the 2-component framed link shown[3] on Fig. 3.2. Through this diffeomorphism K_G goes to $K \subset M \setminus int(N(G))$.

Moreover, L is Kirby-equivalent to the 3-component link N drawn on the right part of Fig. 3.2. It turns out that N is a boundary link. More precisely, up to a (± 1)-framing correction, one can push disjointly N_3, N_1 and then N_2 to the boundary of H_4. We obtain some simple closed curves on $\Sigma_4 = \partial H_4$, which are bounding curves. Therefore, twist along each of these curves induces the identity at the level of $\pi_1(\Sigma_4,*)/\pi_1(\Sigma_4,*)_3$. We then obtain the lemma by a Van-Kampen type argument. □

[3]Blackboard framing convention is used.

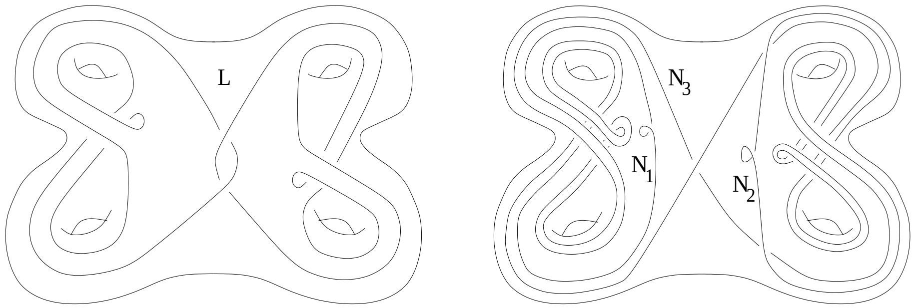

Figure 3.2: A 2-components framed link L associated to a degree 2 clover in H_4, and a Kirby-equivalent boundary link N

3.2. Birman-Craggs homomorphisms for homology cylinders. Birman-Craggs homomorphisms were defined in [**BC**] and they were enumerated in [**J2**]. Levine also outlined in [**Le**] how they can be extended to homology cylinders. In this paragraph, we review Birman-Craggs homomorphisms in a self-contained way. For this, we use the spin refinement of the Goussarov-Habiro theory of finite type invariants: see Chapter 2 of the present work, §2.

We first fix a little notation. If (M,σ) is a closed spin 3-manifold, let $R(M,\sigma) \in \mathbf{Z}_{16}$ denote its Rochlin invariant. If M is a homology sphere, we will denote its (unique) spin structure by σ_0. Recall from Chapter 2 that surgery along a Y-graph makes also sense among spin 3-manifolds:

$$\left(\begin{array}{ll} \textsf{Data:} & \text{(i) } (M,\sigma), \text{ a closed spin 3-manifold} \\ & \text{(ii) } G, \text{ a } Y\text{-graph in } M \end{array} \right) \rightsquigarrow \textsf{Result: } (M_G,\sigma_G).$$

The following lemma describes precisely how the Rochlin invariant is modified during surgery along a Y-graph.

LEMMA 3.4. *Let (M,σ) be a closed spin 3-manifold, and let G be a Y-graph in M whose leaves are ordered, oriented and denoted by K_1, K_2 and K_3. Then,*

$$R(M_G,\sigma_G) - R(M,\sigma) = 8 \cdot \prod_{k=1}^{3} e(t_{K_k})(\sigma) \in \mathbf{Z}_{16}, \tag{3.3}$$

where $8\cdot : \mathbf{Z}_2 \rightarrowtail \mathbf{Z}_{16}$ denotes the usual injection, and where $t_{K_k} \in H_1(E(FM);\mathbf{Z})$ and $e(t_{K_k}) \in A(Spin(M),\mathbf{Z}_2)$ have been defined in §2.2.

PROOF. Let $j : H_3 \hookrightarrow M$ be the embedding of the genus 3 handlebody, determined (up to isotopy) by the Y-graph G in M. Then, it follows from Proposition 3.7 of Chapter 2, that the variation $R(M_G,\sigma_G) - R(M,\sigma)$ only depends on $j^*(\sigma) \in Spin(H_3)$. Also, according to equation (2.1) from the proof of Lemma 2.2, the rhs of (3.3) is determined by $j^*(\sigma) \in Spin(H_3)$.
For $i_1, i_2, i_3 \in \{0,1\}$, we denote by $G_{i_1 i_2 i_3}$ the trivial Y-graph in $\mathbf{S}^3$ (with ordered and oriented leaves) and whose leaf number k is trivial and i_k-framed; we also denote by $j_{i_1 i_2 i_3} : H_3 \hookrightarrow \mathbf{S}^3$ the corresponding embedding. Then,

$$Spin(H_3) = \left\{ j^*_{i_1 i_2 i_3}(\sigma_0) | i_1, i_2, i_3 \in \{0,1\} \right\}.$$

Thus, it is enough to prove (3.3) when (M,σ) is $\left(\mathbf{S}^3,\sigma_0\right)$ and when G is a $G_{i_1 i_2 i_3}$, so that we now restrict ourselves to this case. By Lemma 2.2 b) (iv), the rhs of equation (3.3) is 8 if $i_1 = i_2 = i_3 = 1$ and is 0 otherwise. The same holds for the lhs of equation (3.3). Indeed, surgery along a Y-graph with a trivial leaf has no effect (by the "blow-up move" of [**GGP**, Fig. 6]), and surgery on $\mathbf{S}^3$ along G_{111} gives the Poincaré sphere whose Rochlin invariant is $8 \in \mathbf{Z}_{16}$. It follows that equation (3.3) holds in these eight particular cases. □

Let Σ be Σ_g or $\Sigma_{g,1}$. Let j be an oriented embedding of Σ in $\mathbf{S}^3$, and let $M = (M, i^+, i^-)$ be a homology cylinder over Σ. We can then cut $\mathbf{S}^3$ along $Im(j)$, and glue back M (using the identifications j, i^+ and i^-). We get a new homology sphere which is denoted by

$$\mathbf{S}^3(M, j).$$

It is shown in Corollary 3.8 of Chapter 2 that the Rochlin invariant is a degree 1 invariant: in particular, it is preserved under a Y_2-surgery. Therefore, $R\left(\mathbf{S}^3(M,j), \sigma_0\right)$ only depends on the Y_2-equivalence class of M (and j). Suppose now we are given a surgery presentation of the Y_2-equivalence class of M on 1_Σ:

$$\psi_1\left(\sum_{i=1}^{n} \mathsf{Y}\left[z_1^{(i)}, z_2^{(i)}, z_3^{(i)}\right]\right) = M \in \overline{\mathcal{C}}_1(\Sigma).$$

Recall that the labels $z_k^{(i)}$ belong to P and thus give some $e\left(z_k^{(i)}\right) \in B_g^{(1)}$. We also put $\sigma = j^*(\sigma_0) \in Spin(\Sigma)$, which can be identified with the quadratic form $q_\sigma \in \Omega_g$ according to the Johnson construction (see §2.2). We then deduce from (3.3) the following *cubic* formula:

$$\frac{R\left(\mathbf{S}^3(M,j), \sigma_0\right)}{8} = \sum_{i=1}^{n} \prod_{k=1}^{3} e\left(z_k^{(i)}\right)(q_\sigma) \in \mathbf{Z}_2. \tag{3.4}$$

In particular, this shows that:

(i) $R\left(\mathbf{S}^3(M,j), \sigma_0\right)$ only depends on $\sigma = j^*(\sigma_0) \in Spin(\Sigma)$ (and the Y_2-equivalence class of M);

(ii) if N is another homology cylinder over Σ, then:

$$\frac{R\left(\mathbf{S}^3(M\cdot N,j), \sigma_0\right)}{8} = \frac{R\left(\mathbf{S}^3(M,j), \sigma_0\right)}{8} + \frac{R\left(\mathbf{S}^3(N,j), \sigma_0\right)}{8} \in \mathbf{Z}_2.$$

We now distinguish the case $\Sigma = \Sigma_g$ from the case $\Sigma = \Sigma_{g,1}$.

In the boundary case, any spin structure σ on $\Sigma_{g,1}$ can be realized as a $j^*(\sigma_0)$ for a certain embedding $j : \Sigma_{g,1} \hookrightarrow \mathbf{S}^3$. In fact, the specific embeddings of $\Sigma_{g,1}$ whose images are depicted in Fig. 3.3 do suffice.

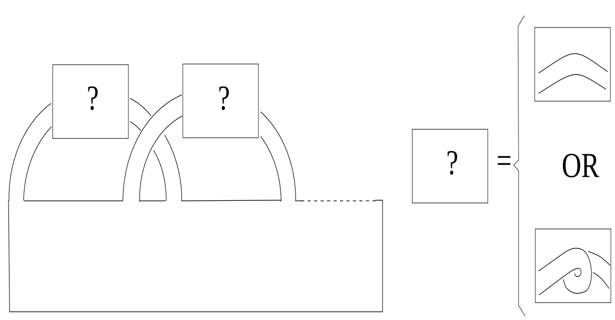

Figure 3.3: Some particular embeddings of $\Sigma_{g,1}$ in $\mathbf{S}^3$

As for the closed case, observe that any embedding $j : \Sigma_g \hookrightarrow \mathbf{S}^3$ is splitting, so that $\sigma = j^*(\sigma_0)$ is spin-bounding. Conversely, any spin structure on Σ_g which spin-bounds can be so realized: choose an appropriate embedding of Σ_g among the particular ones whose images are shown in Fig. 3.4.

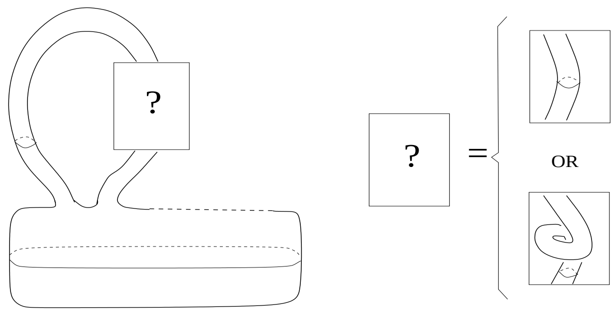

Figure 3.4: Some particular embeddings of Σ_g in $\mathbf{S}^3$

Two other facts about these structures still have to be mentioned. First, $\sigma \in Spin(\Sigma_g)$ spin-bounds if and only if the Arf invariant $\alpha(q_\sigma)$ vanishes (see [**Ki**, p.36]). Second, if f and f' are two cubic polynomials on Ω_g (namely $f, f' \in B_g^{(3)}$), then they are identical on the quadratic forms with trivial Arf invariant if and only if $f - f'$ is a multiple of α (see [**J2**, Lem. 14] for a proof[4] of this algebraic fact).

All of our present discussion leads to the following proposition-definition.

PROPOSITION 3.5. *There exist some well-defined homomorphisms*

$$\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{\beta} B_g^{(3)} \quad \textit{and} \quad \overline{\mathcal{C}}_1(\Sigma_g) \xrightarrow{\beta} \frac{B_g^{(3)}}{\alpha \cdot B_g^{(1)}},$$

such that, for M a homology cylinder over Σ and for $j : \Sigma \hookrightarrow \mathbf{S}^3$ an oriented embedding, we have

$$\beta(M)\left(q_{j^*(\sigma_0)}\right) = \frac{R\left(\mathbf{S}^3(M,j),\sigma_0\right)}{8} \in \mathbf{Z}_2.$$

REMARK 3.6. By composing β with the map $C : \mathcal{T}(\Sigma) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$, we obtain the classical Birman-Craggs homomorphisms, as presented by Johnson in [**J2**].

4. Proof of the results

In this section, we prove the results announced in the introduction.

CONVENTIONS 4.1. In the proofs, we will use some specific techniques of Habiro. Recall that its calculus of claspers developped in [**Hr**] is based on the definition of surgery along a *basic clasper.* So as to be consistent with our Conventions 2.5, we define here a *basic clover C* in a 3-manifold M to be the embedding into M of the surface depicted on the left part of Fig. 4.1. *Surgery along C* is defined as the surgery along the 2-component framed link shown[5] in the right part of Fig. 4.1. Then, a basic clover is a basic clasper but with opposite surgery meaning. Consequently, before using one of the thirteen Habiro's moves, *we will have to take its mirror image.*

[4]There, the proof is given for a genus $g \geq 3$, but the same arguments allow us to prove that this fact also holds for a genus $g = 0, 1$ or 2.

[5]Blackboard framing convention is used.

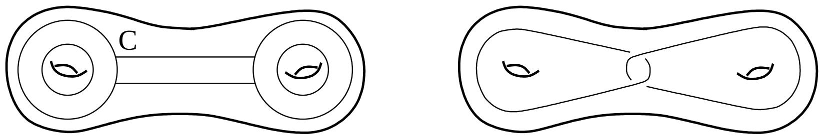

Figure 4.1: A basic clover C and the associated framed link

4.1. Y-equivalence: proof of Proposition 1.1. Since surgeries along clovers preserve homology, the inclusions $\mathcal{C}_1(\Sigma_g) \subset \mathcal{HC}(\Sigma_g)$ and $\mathcal{C}_1(\Sigma_{g,1}) \subset \mathcal{HC}(\Sigma_{g,1})$ are clear.

We now prove the inclusion $\mathcal{HC}(\Sigma_{g,1}) \subset \mathcal{C}_1(\Sigma_{g,1})$ using a result of Habegger. For this, we need the following definition. Let $k \geq 0$ be an integer, a *homology handlebody* of genus k is a pair (M, i) where

(i) M is a compact oriented 3-manifold whose integral homology groups are isomorphic to those of H_k, the standard genus k handlebody;
(ii) $i : \Sigma_k = \partial H_k \longrightarrow M$ is an oriented embedding with image ∂M.

THEOREM 4.2 (Habegger, [**Hg**]). *Let (M_1, i_1) and (M_2, i_2) be genus k homology handlebodies such that*

$$Ker\left(H_1\left(\Sigma_k;\mathbf{Z}\right) \xrightarrow{i_{1,*}} H_1\left(M_1;\mathbf{Z}\right)\right) = Ker\left(H_1\left(\Sigma_k;\mathbf{Z}\right) \xrightarrow{i_{2,*}} H_1\left(M_2;\mathbf{Z}\right)\right).$$

Then, (M_1, i_1) and (M_2, i_2) are Y-equivalent.

In the sequel we identify H_{2g} with $\Sigma_{g,1} \times I$, and so Σ_{2g} with $\partial\left(\Sigma_{g,1 \times I}\right)$. We also denote by (H_{2g}, j) the standard genus $2g$ handlebody, with inclusion $j : \Sigma_{2g} \hookrightarrow H_{2g}$. Any homology cobordism $M = (M, i^+, i^-)$ over $\Sigma_{g,1}$ produces a genus $2g$ homology handlebody (M, i), by defining $i : \Sigma_{2g} \longrightarrow M$ to be the diffeomorphism obtained from the gluing of i^+ with i^-. Suppose now that M is a homology cylinder. Proving that the homology handlebody (M, i) is Y-equivalent to (H_{2g}, j) will imply that the homology cylinder M is Y-equivalent to $(\Sigma_{g,1} \times I, Id, Id)$.

For this, let $x_1^*, \ldots, x_g^*, y_1^*, \ldots, y_g^*$ be some disjoint proper arcs in $\Sigma_{g,1}$, which are "dual" to the loops $x_1, \ldots, x_g, y_1, \ldots, y_g$ of Fig. 3.1, in the sense that x_k^* (resp. y_k^*) transversely intersects x_k (resp. y_k) once but does not intersect the other loops. For example, choose the first attaching region of each 1-handle. For each k, $X_k = x_k^* \times I$ and $Y_k = y_k^* \times I$ are discs in $\Sigma_{g,1} \times I$. The kernel of $j_* : H_1\left(\Sigma_{2g}\right) \longrightarrow H_1\left(\Sigma_{g,1} \times I\right)$ is spanned by $\partial X_1, \ldots, \partial X_g, \partial Y_1, \ldots, \partial Y_g$. On the other hand, observe that $\pm\partial Y_k$ (resp. $\pm\partial X_k$) is homologous to $x_k \times 0 - x_k \times 1$ (resp. to $y_k \times 0 - y_k \times 1$) in Σ_{2g}. Therefore, since M is a homology cylinder, $i\left(\partial X_k\right)$ and $i\left(\partial Y_k\right)$ are nul-homologous in M. As the kernel of $i_* : H_1\left(\Sigma_{2g}\right) \longrightarrow H_1(M)$ has to be of dimension $2g$, it is spanned by $\partial X_1, \ldots, \partial X_g, \partial Y_1, \ldots, \partial Y_g$. It follows from Th. 4.2 that (M, i) is Y-equivalent to (H_{2g}, j), which proves the inclusion $\mathcal{HC}\left(\Sigma_{g,1}\right) \subset \mathcal{C}_1\left(\Sigma_{g,1}\right)$.

Let us now justify the inclusion $\mathcal{HC}\left(\Sigma_g\right) \subset \mathcal{C}_1\left(\Sigma_g\right)$. Let $j : \Sigma_{g,1} \hookrightarrow \Sigma_g$ be an embedding and let $D \subset \Sigma_g$ be its complementary disk. Take a homology cobordism $M = (M, i^+, i^-)$ over $\Sigma_{g,1}$. Then, the embedding $(i^+)|_\partial \circ (j|_\partial)^{-1} = (i^-)|_\partial \circ (j|_\partial)^{-1} : \partial D \hookrightarrow \partial M$ can be stretched to an embedding $\partial D \times I \hookrightarrow \partial M$. The latter allows us to attach the 2-handle $D \times I$ to M. This results in a homology cylinder over Σ_g. We have thus defined a *filling-up* map

$$\mathcal{C}\left(\Sigma_{g,1}\right) \xrightarrow{j} \mathcal{C}\left(\Sigma_g\right),$$

which is obviously surjective. Let $M \in \mathcal{HC}(\Sigma_g)$, and pick a $N \in \mathcal{C}(\Sigma_{g,1})$ such that M is a filling-up of N. Then, N has to be a homology cylinder and so is Y-equivalent to $1_{\Sigma_{g,1}}$. We conclude that $M \in \mathcal{C}_1(\Sigma_g)$, which completes the proof of Proposition 1.1.

4.2. The boundary case: proof of Theorem 1.3. Recall from Example 2.1 that the Abelian group $\mathcal{A}_1(H,0)$ can be identified with $\Lambda^3 H$, and likewise $\mathcal{A}_1\left(H_{(2)},0\right)$ with $\Lambda^3 H_{(2)}$. The following lemma will allow us to identify $\mathcal{A}_1\left(B_g^{(1)},\overline{1}\right)$ with $B_g^{(3)}$.

LEMMA 4.3. *Let $\gamma : \mathcal{A}_1\left(B_g^{(1)},\overline{1}\right) \longrightarrow B_g^{(3)}$ be the map given by multiplying the labels of the abstract Y-graphs: $\gamma(\mathsf{Y}[z_1,z_2,z_3]) = z_1z_2z_3$. Then, γ is a well-defined isomorphism.*

PROOF. The fact that γ is well-defined is clear. In order to show that γ is an isomorphism, it suffices to construct an epimorphism $B_g^{(3)} \xrightarrow{\epsilon} \mathcal{A}_1\left(B_g^{(1)},\overline{1}\right)$ such that $\gamma\circ\epsilon$ is the identity.
By choosing a basis $(e_j)_{j=1}^{2g}$ for H, one determines an isomorphism between $B_g^{(3)}$ and $\mathbf{Z}_2 \oplus H_{(2)} \oplus \Lambda^2 H_{(2)} \oplus \Lambda^3 H_{(2)}$: for $k = 1,2,3$ and $j_1,\ldots,j_k \in \{1,\ldots,2g\}$ pairwise distinct, the monomial $\prod_{i=1}^k \overline{e_{j_i}}$ is identified with the wedge product $\wedge_{i=1}^k e_{j_i}$, and $\overline{1}$ with $1 \in \mathbf{Z}_2$. Since $B_g^{(1)}$ is a period 2 group, so is $\mathcal{A}_1\left(B_g^{(1)},\overline{1}\right)$ by the multilinearity relation. Then, it suffices to define ϵ on the above mentioned $\mathbf{Z}_2$-basis of $\mathbf{Z}_2 \oplus H_{(2)} \oplus \Lambda^2 H_{(2)} \oplus \Lambda^3 H_{(2)} \simeq B_g^{(3)}$. We put $\epsilon(1) = \mathsf{Y}\left[\overline{1},\overline{1},\overline{1}\right]$, $\epsilon(e_j) = \mathsf{Y}\left[\overline{e_j},\overline{1},\overline{1}\right]$, $\epsilon(e_{j_1}\wedge e_{j_2}) = \mathsf{Y}\left[\overline{e_{j_1}},\overline{e_{j_2}},\overline{1}\right]$ (with $j_1 \neq j_2$) and $\epsilon(e_{j_1}\wedge e_{j_2}\wedge e_{j_3}) = \mathsf{Y}\left[\overline{e_{j_1}},\overline{e_{j_2}},\overline{e_{j_3}}\right]$ (with j_1,j_2,j_3 pairwise distinct). The map ϵ is surjective by the multilinearity and slide relations, and obviously satisfies $\gamma\circ\epsilon = Id$. $\square$

Recall from §2.2 that the maps

$$P \xrightarrow{\;p\;} (H,0) \quad \text{and} \quad P \xrightarrow{\;e\;} \left(B_g^{(1)},\overline{1}\right)$$

are the canonical projections of the pullback of special Abelian groups

$$P = (H,0) \times_{(H_{(2)},0)} \left(B_g^{(1)},\overline{1}\right).$$

They happen to be surjective.

LEMMA 4.4. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A}_1(P) \arrow[r, two heads, "\psi_1"] \arrow[dr, "\mathcal{A}_1(p)"'] & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "\eta_1"] \\
 & \mathcal{A}_1(H,0).
\end{tikzcd}$$

PROOF. Let us verify that $\eta_1(\psi_1(Y)) = \mathcal{A}_1(p)(Y)$ for a generator $Y = \mathsf{Y}[z_1,z_2,z_3]$ of $\mathcal{A}_1(P)$. We put $M = \psi_1(Y)$, so that $M = (1_{\Sigma_{g,1}})_G$ where G is an appropriate Y-graph as described in §2.3. Its leaves are in particular ordered and oriented, they are denoted by K_1, K_2 and K_3: $[K_i] = p(z_i) \in H$. Set $\pi = \pi_1(\Sigma_{g,1},*)$ and let $\overline{y} \in \pi/\pi_3$ be represented by $y \in \pi$: we want to compute $\eta_1(M)$ on $\overline{y}$. This goes as follows: choose an immersed based curve k in $\Sigma_{g,1}^+$ representing y (*via* the identification of $\Sigma_{g,1}$ with $\Sigma_{g,1}^+$), pick an oriented based knot $K \subset M$ in a collar of $\Sigma_{g,1}^+$ which is a push-off of k, and find another based knot $K' \subset M$ in a collar of $\Sigma^-_{g,1}$ such that the pairs (M, K) and (M, K') are Y_2-equivalent. Then (*via* the identification of $\Sigma_{g,1}$ with $\Sigma^-_{g,1}$), this knot K' determines a $y' \in \pi$, and by Lemma 3.1, the result $\eta_1(M)(\overline{y})$ is then $\overline{y'} \in \pi/\pi_3$. We now explain the procedure how to construct K' from K.

In $1_{\Sigma_{g,1}} \setminus G$, K can be pushed down in a collar of $\Sigma^-_{g,1}$ *up to* some "fingers" which are of two types (see Fig. 4.2):

(i) the finger is pointing on an edge of G,
(ii) the finger is pointing on an leaf K_i of G.

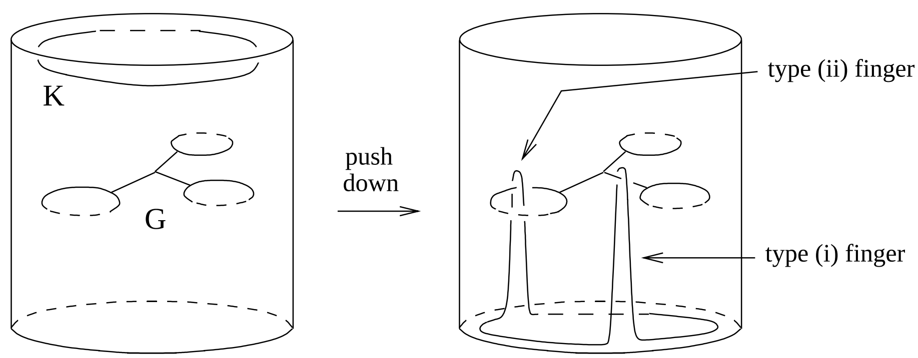

Figure 4.2: Pushing the curve K down the cylinder

But, each finger of type (i) can be isotoped along the corresponding edge towards its leaf and so can be replaced by two fingers of type (ii), so that up to some isotopy of the immersed curve k in $\Sigma^+_{g,1}$, we can suppose each finger to be of type (ii). Since K_i has been oriented, each finger comes with a sign. Let k_i be an immersed curve on $\Sigma^+_{g,1} \subset 1_{\Sigma_{g,1}}$ such that $[k_i] = p(z_i) \in H$. We can suppose that K_i is a push-off of k_i (with possibly an additional twist): there are then as many fingers as intersection points of k_i with k in $\Sigma^+_{g,1}$; the sign of the finger corresponds with the sign of the intersection point contributing to $[k] \bullet [k_i] \in \mathbf{Z}$.
A finger move can be realized by surgery on a basic clover. Let K' be a copy of K in a collar of $\Sigma^-_{g,1} \subset 1_{\Sigma_{g,1}} \setminus G$. There is then a family of basic clovers $\left(C^{(i)}_j\right)^{i=1,2,3}_{j=1,\ldots,n_i}$ in $1_{\Sigma_{g,1}} \setminus G$, such that each $C^{(i)}_j$ has a simple leaf which laces K' and another simple leaf wich laces the leaf K_i, and such that:

$$(M, K) \text{ is diffeomorphic to } \left(1_{\Sigma_{g,1}}, K'\right)_{\left(\cup_{i,j} C^{(i)}_j\right) \cup G}.$$

According to the sign of the corresponding finger, each basic clover comes with a sign denoted by $\varepsilon(i,j)$. *Cutting* the leaf K_1 (see [**GGP**, Cor. 4.3]) n_1 times, we obtain n_1 new Y-graphs $G^{(1)}_j$ ($j \in \{1, \ldots, n_1\}$): two leaves of $G^{(1)}_j$ are copies of K_2 and K_3, and the third leaf forms with a leaf of $C^{(1)}_j$ the Hopf-link. Hence, by applying Habiro's move 2 (or [**GGP**, Th. 2.4]) to $C^{(i)}_j \cup G^{(i)}_j$ we obtain a new Y-graph still denoted by $G^{(i)}_j$. We do the same for $i = 2$ and $i = 3$, therefore:

$$(M, K) \text{ is } Y_2\text{-equivalent to } \left(1_{\Sigma_{g,1}}, K'\right)_{\left(\cup_{i,j} G^{(i)}_j\right) \cup G}.$$

Up to Y_2-equivalence of the pair $\left(1_{\Sigma_{g,1}}, K'\right)_{G \cup \left(\cup_{i,j} G^{(i)}_j\right)}$, one can suppose that, for each (i,j), the whole of $G^{(i)}_j$ lies in a collar neighborhood of $\Sigma^-_{g,1} \subset 1_{\Sigma_{g,1}}$. We now do the surgery along G, and then along each of the $G_j^{(i)}$: the latter does not modify the 3-manifold M but changes the knot. The new knot we obtain is still denoted by K' and satisfies the announced required properties.

We now calculate the $y' \in \pi$ defined by K'. In view of Habiro's move 10, the contribution of each Y-graph $G_j^{(1)}$ to the modification of K' is in π the commutator $\left[k_2, k_3^{-1}\right]^{\varepsilon(1,j)}$. Therefore, we obtain

$$y' \cdot y^{-1} = \prod_{i \in \mathbf{Z}_3} \left[k_{i+1}, k_{i+2}^{-1}\right]^{[k] \bullet [k_i]} \in \frac{\pi_2}{\pi_3}. \tag{4.1}$$

Then, as a homomorphism $H \longrightarrow \pi_2/\pi_3 = L_2(H)$, $\eta_1(M)$ sends any $h \in H$ to

$$-\sum_{i \in \mathbf{Z}_3} (h \bullet p(z_i)) \cdot [p(z_{i+1}), p(z_{i+2})] \in L_2(H).$$

which corresponds to $\sum_{i \in \mathbf{Z}_3} p(z_i) \otimes [p(z_{i+1}, p(z_{i+2})]$ in $H \otimes L_2(H)$, to $p(z_1) \wedge p(z_2) \wedge p(z_3)$ in $\Lambda^3 H$, and so to $\mathcal{A}_1(p)(Y)$. □

LEMMA 4.5. *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{A}_1(P) \arrow[r, twoheadrightarrow, "\psi_1"] \arrow[dr, "\mathcal{A}_1(e)"'] & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "\beta"] \\ & \mathcal{A}_1\left(B_g^{(1)}, \overline{1}\right). \end{tikzcd}$$

PROOF. According to the definition of β we gave in Proposition 3.5, this is a direct consequence of equation (3.4). □

We still denote by

$$(H, 0) \xrightarrow{- \otimes \mathbf{Z}_2} \left(H_{(2)}, 0\right) \text{ and } \left(B_g^{(1)}, \overline{1}\right) \xrightarrow{\kappa} \left(H_{(2)}, 0\right),$$

the maps which appear in the pullback diagram for P (see §2.2). Then, as a consequence of the two preceeding lemmas, $\mathcal{A}_1(\kappa)\beta\psi_1 = \mathcal{A}_1(\kappa e) = \mathcal{A}_1\left((- \otimes \mathbf{Z}_2)p\right) = \mathcal{A}_1(- \otimes \mathbf{Z}_2)\eta_1\psi_1$. Since ψ_1 is an epimorphism, we get: $\mathcal{A}_1(\kappa)\beta = \mathcal{A}_1(- \otimes \mathbf{Z}_2)\eta_1$. Construct the following pull-back:

$$\begin{tikzcd} \mathcal{A}_1(H,0) \times_{\mathcal{A}_1(H_{(2)},0)} \mathcal{A}_1\left(B_g^{(1)}, \overline{1}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}_1\left(B_g^{(1)}, \overline{1}\right) \arrow[d, "\mathcal{A}_1(\kappa)"] \\ \mathcal{A}_1(H,0) \arrow[r, "\mathcal{A}_1(- \otimes \mathbf{Z}_2)"'] & \mathcal{A}_1(H_{(2)},0) \end{tikzcd}$$

which, through the above mentioned identifications, is essentially the pull-back diagram for $\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ appearing in §1.3. By the universal property of the pull-backs, there is then a homomorphism

$$\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{(\eta_1, \beta)} \mathcal{A}_1(H,0) \times_{\mathcal{A}_1(H_{(2)},0)} \mathcal{A}_1\left(B_g^{(1)}, \overline{1}\right) \simeq \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}.$$

Moreover, we also have by functoriality another natural map

$$\mathcal{A}_1\big(\underbrace{(H,0) \times_{(H_{(2)},0)} (B_g^{(1)}, \overline{1})}_{P}\big) \xrightarrow{\rho} \mathcal{A}_1(H,0) \times_{\mathcal{A}_1(H_{(2)},0)} \mathcal{A}_1\left(B_g^{(1)}, \overline{1}\right).$$

Lemma 4.4 and Lemma 4.5 can then be summarized in the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{A}_1(P) \arrow[r, two heads, "\psi_1"] \arrow[dr, "\rho"'] & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "{(\eta_1,\beta)}"] \\
& \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}.
\end{tikzcd}$$

The following lemma will be the final step in proving Theorem 1.3.

LEMMA 4.6. *The map* $\rho : \mathcal{A}_1(P) \longrightarrow \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ *is an isomorphism.*

Assume Lemma 4.6. Then, from the previous commutative diagram, it follows that ψ_1 is injective, and so is an isomorphism: as a consequence, the same holds for (η_1, β). The commutativity of

$$\begin{tikzcd}
\overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "{(\eta_1,\beta)}"', "\simeq"] & \frac{\mathcal{T}_{g,1}}{\mathcal{T}'_{g,1}} \arrow[l, "C"'] \arrow[dl, "{(\eta_1,\beta)}"] \\
\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} &
\end{tikzcd}$$

follows from Remark 3.3 and Remark 3.6. In particular, when $g \geq 3$, C is an isomorphism because $(\eta_1, \beta) : \mathcal{T}_{g,1}/\mathcal{T}'_{g,1} \longrightarrow \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ is so by [**J4**].

PROOF OF LEMMA 4.6. We proceed as in Lemma 4.3. It suffices to construct an epimorphism

$$\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} \overset{\epsilon}{\twoheadrightarrow} \mathcal{A}_1(P)$$

such that $\rho \circ \epsilon$ is the identity.
Pick a basis $(e_i)_{i=1}^{2g}$ of H: we have seen in the proof of Lemma 4.3 that this choice determines an isomorphism between $B_g^{(3)}$ and $\Lambda^3 H_{(2)} \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$. Thus, it also defines an isomorphism between $\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ and $\Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$. We now define ϵ by putting

(i) $\epsilon(e_i \wedge e_j \wedge e_k) = \mathsf{Y}\left[(e_i, \overline{e_i}), (e_j, \overline{e_j}), (e_k, \overline{e_k})\right]$, with $1 \leq i < j < k \leq 2g$,
(ii) $\epsilon(e_i \wedge e_j) = \mathsf{Y}\left[(e_i, \overline{e_i}), (e_j, \overline{e_j}), (0, \overline{1})\right]$, with $1 \leq i < j \leq 2g$,
(iii) $\epsilon(e_i) = \mathsf{Y}\left[(e_i, \overline{e_i}), (0, \overline{1}), (0, \overline{1})\right]$, with $1 \leq i \leq 2g$,
(iv) and $\epsilon(1) = \mathsf{Y}\left[(0, \overline{1}), (0, \overline{1}), (0, \overline{1})\right]$.

Here, elements of P are denoted as in Remark 2.3. This assignation well defines ϵ because (i) determines ϵ on a basis of the free group $\Lambda^3 H$, while (ii),(iii) and (iv) assign elements of $\mathcal{A}_1(P)$ of order at most 2 to each element basis of the $\mathbf{Z}_2$-vector space $\Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$. Obviously, ϵ followed by ρ gives the identity. Take now any generator $\mathsf{Y}[z_1, z_2, z_3]$ of $\mathcal{A}_1(P)$. For $i = 1, 2, 3$, $z_i \in P$ can be written as a linear combination of some $(e_j, \overline{e_j})$ and $(0, \overline{1})$. The multilinearity, AS and slide relation allow us to conclude that $\mathsf{Y}[z_1, z_2, z_3]$ is realized by ϵ. Thus, ϵ is surjective. □

4.3. The closed case: proof of Theorem 1.4. An isomorphism

$$\mathcal{A}_1(P) \xrightarrow{\rho} \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$$

is defined formally in the same way as in the boundary case (see Lemma 4.6). Recall that S stands for the subgroup of the pullback $\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ corresponding to

$\omega \wedge H \subset \Lambda^3 H$ and $\alpha \cdot B_g^{(1)} \subset B_g^{(3)}$. Then, $\rho^{-1}(\mathrm{S})$ is the subgroup of $\mathcal{A}_1(P)$ comprising the elements

$$\sum_{i=1}^{g} \mathsf{Y}\left[(x_i, \overline{x_i}), (y_i, \overline{y_i}), z\right], \quad \text{where } z \text{ is any element of } P.$$

LEMMA 4.7. *In the closed case, the surgery map ψ_1 defined in §2.3 vanishes on the subspace $\rho^{-1}(S)$.*

As mentioned in the introduction, these symplectic relations $\rho^{-1}(\mathrm{S})$ appears in [**Hr**] for higher degrees.

PROOF OF LEMMA 4.7. Let $z \in P$, we aim to show that

$$\sum_{i=1}^{g} \psi_1\left(\mathsf{Y}\left[(x_i, \overline{x_i}), (y_i, \overline{y_i}), z\right]\right) = 0 \in \overline{\mathcal{C}}_1(\Sigma_g). \tag{4.2}$$

Consider in 1_{Σ_g} a basic clover G with one trivial leaf f, and the other leaf f' satisfying $t_{f'} = z \in P$. Then, f being trivial, $\left(1_{\Sigma_g}\right)_G$ is diffeomorphic to 1_{Σ_g}. Furthermore, f can be seen as a push-off of ∂D where D is a 2-disk in Σ_g^+: in particular, f bounds the push-off of $\Sigma_g^+ \setminus D$ which is an embedded genus g surface. By applying Habiro's moves 7 and 5, f can be split in g pieces so that G is equivalent to the union of g basic clovers denoted by $G_1, \ldots, G_g$. See Fig. 4.3. Each clover G_i

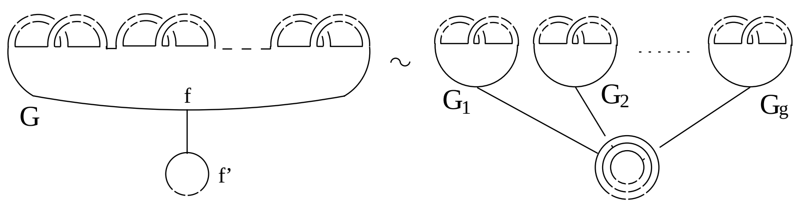

Figure 4.3: Splitting the null-homologous leaf f

has a leaf which bounds a genus 1 surface; by applying Habiro's move 10, it is seen to be equivalent to a Y-graph G_i'. According to Remark 2.3, the leaves of G_i' represent $(x_i, \overline{x_i})$, $(y_i, \overline{y_i})$ and z in P, so that $\left(1_{\Sigma_g}\right)_{G_i'} = \psi_1\left(\mathsf{Y}\left[(x_i, \overline{x_i}), (y_i, \overline{y_i}), z\right]\right) \in \overline{\mathcal{C}}_1(\Sigma_g)$. Equation (4.2) then follows. □

By the same arguments, appropriate versions of Lemma 4.4 and Lemma 4.5 hold in the closed case: $\mathcal{A}_1(p) = \eta_1 \circ \psi_1$ and $\mathcal{A}_1(e) = \beta \circ \psi_1$. This leads us to a commutative diagram

$$\begin{tikzcd}
\dfrac{\mathcal{A}_1(P)}{\rho^{-1}(\mathrm{S})} \arrow[r, "\psi_1"] \arrow[dr, "\simeq", "\rho"'] & \overline{\mathcal{C}}_1(\Sigma_g) \arrow[d, "(\eta_1, \beta)"] \\
 & \dfrac{\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}}{\mathrm{S}} \simeq \dfrac{\Lambda^3 H}{\omega \wedge H} \times_{\left(\frac{\Lambda^3 H_{(2)}}{\omega_{(2)} \wedge H_{(2)}}\right)} \dfrac{B_g^{(3)}}{\alpha \cdot B_g^{(1)}}
\end{tikzcd}$$

from which it follows that ψ_1, and then (η_1, β), are isomorphisms. The commutativity of the right triangle in Th. 1.4 is still given by Rem. 3.3 and Rem. 3.6.

4.4. Finite type invariants of degree 1: proof of Corollary 1.5. The equivalence (a)$\Leftrightarrow$(b) immediately results from the existence of the universal degree one additive invariant v introduced in Remark 2.7. The equivalence (c)$\Leftrightarrow$(a) is a direct consequence of Theorem 1.4 and Theorem 1.3.

4.5. From the boundary case to the closed case. In this last paragraph, we *fix* an isomorphism

$$H_1(\Sigma_{g,1};\mathbf{Z}) \xrightarrow{\phi} H_1(\Sigma_g;\mathbf{Z}).$$

It allows us to identify the sets H, $\Omega_g \simeq Spin(\Sigma)$, B_g and P corresponding to $\Sigma_{g,1}$ with those of Σ_g.
Moreover, let $j:\Sigma_{g,1} \hookrightarrow \Sigma_g$ be an embedding such that $j_* = \phi$ at the level of $H_1(-;\mathbf{Z})$. Recall from §4.1 the filling-up map, which can be restricted to

$$\mathcal{C}_1\left(\Sigma_{g,1}\right) \xrightarrow{j} \mathcal{C}_1\left(\Sigma_g\right).$$

Note that it is compatible with the "extending by the identity" map $\mathcal{T}_{g,1} \longrightarrow \mathcal{T}_g$ defined by j, and that it induces a group homomorphism $\overline{\mathcal{C}}_1\left(\Sigma_{g,1}\right) \longrightarrow \overline{\mathcal{C}}_1\left(\Sigma_g\right)$. The latter can be verified to be independent on the choice of the embedding j such that $j_* = \phi$, and so can be denoted by

$$\overline{\mathcal{C}}_1\left(\Sigma_{g,1}\right) \xrightarrow{\phi} \overline{\mathcal{C}}_1\left(\Sigma_g\right).$$

The commutativity of the following diagram is easily proved from the various definitions:

$$\begin{tikzcd}
\mathcal{A}_1(P) \arrow[rr, "\psi_1"] \arrow[dr, "\rho"'] \arrow[dd] & & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[dl, "(\eta_1,\beta)"'] \arrow[dd, "\phi"] & \\
& \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} \arrow[dd] & & \dfrac{\mathcal{T}_{g,1}}{\mathcal{T}'_{g,1}} \arrow[ul, "C"'] \arrow[ll, "(\eta_1,\beta)"] \arrow[dd] \\
\dfrac{\mathcal{A}_1(P)}{\rho^{-1}(\mathrm{S})} \arrow[rr, "\psi_1"] \arrow[dr, "\rho"'] & & \overline{\mathcal{C}}_1(\Sigma_g) \arrow[dl, "(\eta_1,\beta)"'] & \\
& \dfrac{\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}}{\mathrm{S}} & & \dfrac{\mathcal{T}_g}{\mathcal{T}'_g} \arrow[ul, "C"'] \arrow[ll, "(\eta_1,\beta)"]
\end{tikzcd}$$

CHAPITRE 4

Fonctions quadratiques et invariants de type fini des $Spin^c$-variétés fermées de dimension 3

En collaboration avec Florian Deloup.

1. Introduction

Ever since the beginning of algebraic topology, quadratic functions have played a prominent role. This chapter is an attempt to study systematically the role of quadratic functions in 3-dimensional topology, at least in the case of closed oriented 3-manifolds.

The first part (§2) of the chapter is entirely algebraic and is devoted to the study of quadratic functions on torsion Abelian groups. A quadratic function on a torsion Abelian group G is a function $q : G \to \mathbb{Q}/\mathbb{Z}$ such that the map $b_q : (x, y) \mapsto q(x + y) - q(x) - q(y)$ is bilinear. If b_q is a nondegenerate bilinear pairing, we say that q is a nondegenerate quadratic function. Because of 2-torsion in G, a quadratic function q cannot be recovered from its associated bilinear pairing b_q in general. Therefore, the theory of quadratic functions is finer than the theory of symmetric bilinear pairings.
For finite Abelian groups, only the latter are classified ([**W1**] over odd p-groups, and [**KK**] over 2-groups). Note that our definition includes quadratic functions that are not homogeneous, that is, that do not satisfy $q(x) = q(-x)$. It is therefore natural, for a quadratic function q, to consider the homogeneity defect $d_q \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$, defined by $d_q(x) = q(x) - q(-x)$. When G is finite, an invariant of the isomorphism class of q is provided by its Gauss sum, which is defined by $\gamma(q) = |G|^{-1/2} \sum_{x \in G} e^{2\pi i q(x)} \in \mathbb{C}$. Our main result is the following one.

Theorem 1. *Let $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ be two nondegenerate quadratic functions on finite Abelian groups. Then q is isomorphic to q' if and only if there is an isomorphism $\psi : G \to G'$ such that $b_q = b_{q'} \circ \psi^{\otimes 2}$, $d_q = d_{q'} \circ \psi$ and $\gamma(q) = \gamma(q')$.*

Turning to topology in §3, we then consider closed oriented 3-manifolds and some additional structures with which they can be endowed, called complex spin structures or Spin^c-structures. They are related to spin structures in the sense that if M is a closed oriented 3-manifold, there is then a canonical map $\mathrm{Spin}(M) \to \mathrm{Spin}^c(M)$, from the space of its Spin-structures to the space of its Spin^c-structures. Complex spin structures are also worth to be taking into account in 3-dimensional topology, at least because they are in natural correspondance with so-called Euler structures, which have been introduced by Turaev in order to refine Reidemeister torsions.

We give here a general homotopy-theoritical treatment of Spin^c-structures (following that of Spin-structures given in [**BM**]). Having in mind the importance of "cut and paste" techniques in 3-dimensional topology, we deal with the technical problem of gluing Spin^c-structures. For that aim, we define for any compact oriented 3-manifold with boundary and any Spin-structure σ on ∂M, a space denoted by $\mathrm{Spin}^c(M,\sigma)$, of complex spin structures on M relative to σ (see Theorem 3.10). We also describe how Spin^c-manifolds can be presented by surgery on $\mathbf{S}^3$ and we give the corresponding Spin^c-refinement of Kirby's theorem.

We then relate in §4 algebra to topology, quadratic functions to closed Spin^c-manifolds of dimension 3. By means of that Spin^c-refinement of Kirby's theorem, we associate to an arbitrary closed Spin^c-manifold (M,σ) of dimension 3 a quadratic function

$$\phi_{M,\sigma} : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z},$$

whose associated bilinear pairing, denoted by $L_M : H_2(M;\mathbb{Q}/\mathbb{Z}) \times H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$, is a slight modification of the torsion linking pairing of M. Explicitly, let $\lambda_M : \mathrm{T}H_1(M) \times \mathrm{T}H_1(M) \to \mathbb{Q}/\mathbb{Z}$ be the torsion linking pairing of M, where $\mathrm{T}H_1(M)$ denotes the torsion subgroup of M, then $L_M = \lambda_M \circ B^{\otimes 2}$, where $B : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathrm{T}H_1(M)$ is the Bockstein homomorphism associated to the short exact sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Note that L_M (and therefore $\phi_{M,\sigma}$) is nondegenerate if and only if M is a rational homology 3-sphere. When σ comes from a spin structure, the quadratic function $\phi_{M,\sigma}$ turns out to be equivalent with previous works ([**LL**], [**MS**], [**T2**]). More generally, when σ is torsion (which means that its Chern class $c(\sigma) \in H^2(M)$ is torsion), $\phi_{M,\sigma}$ factorizes *via* B to a quadratic function:

$$\phi_{M,\sigma} : \mathrm{T}H_1(M) \to \mathbb{Q}/\mathbb{Z},$$

over the torsion linking pairing λ_M and then coincides with analogous constructions from [**Gi**] (see also [**De3**]) and [**LW**].

We then show that for a given closed oriented 3-manifold M, the Spin^c-structures on M are determined by their corresponding quadratic functions.

THEOREM 2. *Let M be a closed connected oriented 3-manifold. Then, the map $[\sigma \mapsto \phi_{M,\sigma}]$ defines an affine embedding:*

$$\mathrm{Spin}^c(M) \xhookrightarrow{\Phi_M} \mathrm{Quad}(L_M).$$

We also identify the cokernel of ϕ_M (Theorem 4.10 and §4.3). *Via* the map ϕ_M, topological notions can be put in correspondance with algebraic notions. For instance, the Chern class $c(\sigma)$ corresponds with the homogeneity defect $d_{\phi_{M,\sigma}}$ of the quadratic function $\phi_{M,\sigma}$.
When σ is identified with an Euler structure, a geometric intrinsic formula (making no reference to the dimension 4) for the quadratic function $\phi_{M,\sigma}$ is given. When M is a rational homology sphere, $\phi_{M,\sigma}$ can be recovered from the Reidemeister-Turaev torsion (Theorem 4.17).

In the final part (§5) of the paper, we apply the previous to solve a problem related to the Goussarov-Habiro theory for compact oriented 3-manifolds. Their finite type invariants theory (developped in [**Go**], [**Hr**] and [**GGP**]) is based on an elementary move called Y-surgery. The Y-equivalence, which is defined to be the surgery equivalence relation generated by this move, is then of crucial and basic importance in the Goussarov-Habiro theory, since two manifolds are Y-equivalent if and only if they are not distinguished by degree 0 finite type invariants. It has been characterized in the closed case by Matveev, who showed in [**Mt**] that two closed oriented connected 3-manifolds are Y-equivalent if and only if they have identical first Betti numbers and isomorphic torsion linking pairings. We have introduced in Chapter 2 a non-trivial Spin-refinement of the Goussarov-Habiro theory and we have characterized degree 0 invariants of Spin-manifolds by refining that theorem of Matveev.

Using the gluing techniques developped in §3, we show that Y-surgeries make also sense for Spin^c-manifolds, and generate a surgery equivalence relation among them called Y^c-equivalence. There exists then a Spin^c-refinement of the Goussarov-Habiro theory, the degree 0 invariants of which are characterized in the closed case by the following refinement of Matveev's theorem.

THEOREM 3. *Let (M,σ) and (M',σ') be two closed connected oriented 3-manifolds with complex spin structures. The following assertions are then equivalent:*

1. *the Spin^c-manifolds (M,σ) and (M',σ') are Y^c-equivalent;*
2. *there is an isomorphism $\psi : H_1(M) \to H_1(M')$ such that $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \psi^\sharp$;*
3. *there is an isomorphism $\psi : H_1(M) \to H_1(M')$ such that:*
   - $\lambda_M = \lambda_{M'} \circ (\psi|)^{\otimes 2}$,
   - $\psi\left(P^{-1}c(\sigma)\right) = P^{-1}c\left(\sigma'\right)$,
   - $\gamma\left(\phi_{M,\sigma} \circ s\right) = \gamma\left(\phi_{M',\sigma'} \circ s'\right)$ *for any pair (s,s') of compatible sections of the respective Bockstein homomorphisms $B : H_2\left(M;\mathbb{Q}/\mathbb{Z}\right) \to \mathrm{T}H_1(M)$ and $B : H_2\left(M';\mathbb{Q}/\mathbb{Z}\right) \to \mathrm{T}H_1(M')$.*

In assertion (2), $\psi^\sharp : H_2\left(M';\mathbb{Q}/\mathbb{Z}\right) \to H_2(M;\mathbb{Q}/\mathbb{Z})$ is the isomorphism dual to ψ by the intersection pairings:

$$H_1(M) \times H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad H_1(M') \times H_2(M';\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z},$$

which are right nonsingular. In assertion (3), P stands for a Poincaré duality isomorphism, and the required compatibility of sections s, s' is the commutativity of the diagram:

$$\begin{tikzcd} H_2\left(M';\mathbb{Q}/\mathbb{Z}\right) \arrow[d, "\psi^\sharp"', "\simeq"] & \mathrm{T}H_1(M') \arrow[l, "s'"'] \\ H_2\left(M;\mathbb{Q}/\mathbb{Z}\right) & \mathrm{T}H_1(M). \arrow[l, "s"] \arrow[u, "\psi|", "\simeq"'] \end{tikzcd}$$

The equivalence between assertions (2) and (3) will be derived from Theorem 1.

The case of rational homology 3-spheres deserves to be singled out. Indeed, if M is an oriented rational homology 3-sphere, any $\sigma \in \mathrm{Spin}^c(M)$ is then torsion and, according to what has been said above, $\phi_{M,\sigma}$ can be regarded as a quadratic function $H_1(M) \to \mathbb{Q}/\mathbb{Z}$ over λ_M. In that case, Theorem 3 specializes to:

COROLLARY 1. *Let (M,σ) and (M',σ') be two oriented rational homology 3-spheres with complex spin structures. The following assertions are then equivalent:*

1. *the Spin^c-manifolds (M,σ) and (M',σ') are Y^c-equivalent;*
2. *there is an isomorphism $\psi : H_1(M) \to H_1(M')$ such that $\phi_{M,\sigma} = \phi_{M',\sigma'} \circ \psi$;*
3. *there is an isomorphism $\psi : H_1(M) \to H_1(M')$ such that:*
   - $\lambda_M = \lambda_{M'} \circ \psi^{\otimes 2}$,
   - $\psi\left(P^{-1}c(\sigma)\right) = P^{-1}c(\sigma')$,
   - $\gamma(\phi_{M,\sigma}) = \gamma(\phi_{M',\sigma'})$.

It follows also from Theorem 3 and from Theorem 1.1 of Chap. 2 that, although the canonical map $\mathrm{Spin}(M) \to \mathrm{Spin}^c(M)$ needs not be injective, the Spin-refinement of Goussarov-Habiro theory *embeds in degree* 0 into its Spin^c-refinement.

2. Quadratic functions on torsion Abelian groups

Let us fix a few conventions. Given an Abelian group M, $\mathrm{T}M$ denotes the subgroup of M consisting of all torsion elements. If M and N are two Abelian groups, $\mathrm{Hom}(M,N)$ denotes the Abelian group of all homomorphisms $M \to N$. A set X on which M acts freely and transitively is an *affine space over* M. For such actions, the left multiplicative notation is used. Any bilinear pairing $b : M \times M' \to N$, where M, M' and N are Abelian groups, has a left (resp. right) adjoint map ${}_\bullet\widehat{b} : M \to \mathrm{Hom}(M',N)$ (resp. $\widehat{b}_\bullet : M' \to \mathrm{Hom}(M,N)$), defined by ${}_\bullet\widehat{b}(x)(y) = b(x,y)$ (resp. $\widehat{b}_\bullet(y)(x) = b(x,y)$), $x \in M$, $y \in M'$. In the case when $M = M'$ and b is symmetric, ${}_\bullet\widehat{b} = \widehat{b}_\bullet$ is denoted by $\widehat{b}$.

If M is a free Abelian group, we set $C_M = \mathbb{Z}$; if M is a $\mathbb{Q}$-vector space, we set $C_M = \mathbb{Q}$; if M is a torsion Abelian group, we set $C_M = \mathbb{Q}/\mathbb{Z}$. In such cases, it is convenient to define the dual of M by $M^* = \mathrm{Hom}(M, C_M)$.

2.1. Basic notions about quadratic functions. Let M and N be Abelian groups. A *quadratic function* on M with values in N is a map $q : M \to N$ such that the formula $b_q(x,y) = q(x+y) - q(x) - q(y)$, $x, y \in M$, defines a symmetric bilinear pairing $b_q : M \times M \to N$, called the *associated bilinear pairing.* Note that according to the definition we adopt in this chapter, any quadratic function q satisfies $q(0) = 0$. In the case when q satisfies the additional property that $q(rx) = r^2 q(x)$ for all $x \in M$, $r \in \mathbb{Z}$, we say that q is *homogeneous.* To each quadratic function q, we associate an element $d_q \in \mathrm{Hom}(M,N)$, called the *homogeneity defect* and defined by $d_q(x) = q(x) - q(-x)$, $x \in M$. Clearly, q is homogeneous if and only if $d_q = 0$. The quadratic function q is *nondegenerate* (resp. *nonsingular* or *unimodular*) if the adjoint map $\widehat{b_q} : M \to \mathrm{Hom}(M,N)$ is injective (resp. bijective). If M is a finite Abelian group and $N = C_M = \mathbb{Q}/\mathbb{Z}$, q is nondegenerate if and only if q is nonsingular.

A quadratic function $q : M \to N$ is a quadratic function *over* $b : M \times M \to N$ if $b_q = b$. Denote by $\mathrm{Quad}(b)$ the set of quadratic functions over b. The Abelian group $\mathrm{Hom}(M,N)$ acts freely and transitively on $\mathrm{Quad}(b)$ by addition. Thus, if b is nonsingular then $\mathrm{Quad}(b)$ is an affine space over M with action defined by:

$$\alpha \cdot q = q + \widehat{b}(\alpha), \ \alpha \in M.$$

It follows that, in that case, $\mathrm{Quad}(b)$ and M are equipotent.

Let $\psi : M' \to M$ be an isomorphism of Abelian groups. We define $\psi^* b : M' \times M' \to N$ to be the symmetric bilinear pairing defined by $\psi^* b(x,y) = b(\psi(x), \psi(y))$ for all $x, y \in M'$. Similarly we define the quadratic function $\psi^* q : M' \to N$ by $\psi^* q(x) = q(\psi(x))$ for all $x \in M'$. We say that two symmetric bilinear pairings b, b' (resp. two quadratic functions q, q') defined on M and M' with values in N are *isomorphic*, and we write $b \sim b'$ (resp. $q \sim q'$), if there exists an isomorphism $\psi : M' \to M$ such that $\psi^* b = b'$ (resp. $\psi^* q = q'$). An isomorphism $\psi : M' \to M$ induces a bijective correspondence between $\mathrm{Quad}(b)$ and $\mathrm{Quad}(\psi^* b)$. In particular, the subgroup $\mathrm{Iso}(b)$ of automorphisms of M preserving b acts on $\mathrm{Quad}(b)$: $\psi \cdot q = \psi^* q$. The relation between the action of $\mathrm{Hom}(M,N)$ and the action of $\mathrm{Iso}(b)$ on $\mathrm{Quad}(b)$ is the key to the problem of classification of quadratic functions.

2.2. Presentations of quadratic functions.

2.2.1. *Lattices and their characteristic forms.* A *lattice* M is a finitely generated free Abelian group. A *(symmetric) bilinear lattice* (M, f) is a symmetric bilinear form $f : M \times M \to \mathbb{Z}$ on a lattice M. Then $M \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector space which we denote by V. Any bilinear lattice (M, f) gives rise by extension of scalars to a bilinear pairing $f_{\mathbb{Q}} : V \times V \to \mathbb{Q}$. Let $M^\sharp = \{x \in V : f_{\mathbb{Q}}(x, M) \subset \mathbb{Z}\}$ be the *dual lattice* for (M, f). Clearly $M \subset M^\sharp$. More generally, for any subgroup N of V, we can define $N^\sharp = \{x \in V : f_{\mathbb{Q}}(x, N) \subset \mathbb{Z}\}$.
A *fractional* (resp. *integral*) *Wu class* for (M, f) is an element $v \in V$ (resp. an element $v \in M$) such that:

$$\forall x \in M,\ f(x,x) - f_{\mathbb{Q}}(w, x) \in 2\mathbb{Z}.$$

The set of fractional (resp. integral) Wu classes is denoted by $\mathrm{Wu}^{\mathbb{Q}}(f)$ (resp. $\mathrm{Wu}(f)$) and is contained in $M^\sharp$. Furthermore, $\mathrm{Wu}^{\mathbb{Q}}(f)$ is an affine space over $2M^\sharp$.
A *characteristic form* for f is an element $c \in M^*$ satisfying:

$$\forall x \in M,\ f(x,x) - c(x) \in 2\mathbb{Z}.$$

The set $\mathrm{Char}(f)$ of characteristic forms for f is an affine space over $\mathrm{Hom}(M, 2\mathbb{Z})$. Each fractional Wu class gives a characteristic form by the equivariant map $w \mapsto f_{\mathbb{Q}}(w, -)|_M$, $\mathrm{Wu}^{\mathbb{Q}}(f) \to \mathrm{Char}(f)$.

LEMMA 2.1. *If f is nondegenerate, then the map $w \mapsto f_{\mathbb{Q}}(w, -)|_M$ is a bijective correspondence between $\mathrm{Wu}^{\mathbb{Q}}(f)$ and $\mathrm{Char}(f)$.*

PROOF. f nondegenerate means $\widehat{f} : M \to M^*$ is injective. Then $\widehat{f_{\mathbb{Q}}} : V \to V^*$ is bijective. Hence, the map $x \mapsto f_{\mathbb{Q}}(x, -)|_M$, $\mathrm{Wu}^{\mathbb{Q}}(f) \to \mathrm{Char}(f)$ is also bijective. □

The quotient group $\bar{M} = M/\mathrm{Ker}\ \widehat{f}$ is a finitely generated free Abelian group. Hence, the short exact sequence:

$$0 \longrightarrow \mathrm{Ker}\ \widehat{f} \xrightarrow{\ i\ } M \xrightarrow{\ p\ } \bar{M} \longrightarrow 0 \tag{2.1}$$

is split. Any section s of p then induces a (non-canonical) isomorphism:

$$(M, f) \simeq (\bar{M}, \bar{f}) \oplus \left(\mathrm{Ker}\ \widehat{f}, 0\right),$$

where $\bar{f} : \bar{M} \times \bar{M} \to \mathbb{Z}$ is the nondegenerate pairing induced by f.

LEMMA 2.2. *There is an injection $p^*|_{\mathrm{Char}(\bar{f})} : \mathrm{Char}\left(\bar{f}\right) \to \mathrm{Char}(f)$ induced by p, and any section s of p induces a retraction $s^*|_{\mathrm{Char}(f)} : \mathrm{Char}(f) \to \mathrm{Char}(\bar{f})$ for it.*

PROOF. Since $p \circ s = \mathrm{Id}_{\bar{M}}$, we have: $s^* \circ p^* = \mathrm{Id}_{\bar{M}^*}$. Moreover, one easily verifies that: $p^*\left(\mathrm{Char}\left(\bar{f}\right)\right) \subset \mathrm{Char}\left(f\right)$, and that: $s^*\left(\mathrm{Char}\left(f\right)\right) \subset \mathrm{Char}\left(\bar{f}\right)$. □

LEMMA 2.3. *Every symmetric bilinear lattice (M, f) has a characteristic form.*

PROOF. According to Lemma 2.2, we can suppose that f is nondegenerate. In the nondegenerate case, it is shown in [**LW**, Lemma 1.6 (i)] that any (M, f) admits an integral Wu class. □

2.2.2. *The discriminant construction.* Suppose we are given a bilinear lattice (M, f) as above. Consider the torsion Abelian group $G_f = M^\sharp/M$ and the symmetric bilinear pairing

$$L_f : G_f \times G_f \to \mathbb{Q}/\mathbb{Z}$$

defined by:

$$L_f\left([x],[y]\right) = f_{\mathbb{Q}}(x,y) \bmod \mathbb{Z}, \quad x, y \in M^\sharp. \tag{2.2}$$

Since $M^{\sharp\sharp} = M + \operatorname{Ker} \widehat{f_{\mathbb{Q}}}$, the radical of L_f is:

$$\operatorname{Ker} \widehat{L_f} = \left(M + \operatorname{Ker} \widehat{f_{\mathbb{Q}}}\right)/M = \operatorname{Ker} \widehat{f_{\mathbb{Q}}}/\operatorname{Ker} \widehat{f} = \left(\operatorname{Ker} \widehat{f}\right) \otimes \mathbb{Q}/\mathbb{Z}. \tag{2.3}$$

In particular, L_f is nondegenerate if and only if f is nondegenerate.
Consider the torsion subgroup T Coker $\widehat{f}$ of Coker $\widehat{f}$. The adjoint map $\widehat{f_{\mathbb{Q}}} : V \to V^*$ restricted to $M^\sharp$ induces a canonical epimorphism $B_f : G_f \to \mathrm{T}$ Coker $\widehat{f}$. Hence the following short exact sequence:

$$0 \to \operatorname{Ker} \widehat{L_f} \to G_f \to \mathrm{T} \operatorname{Coker} \widehat{f} \to 0. \tag{2.4}$$

It follows that the symmetric bilinear form L_f factorizes to a nondegenerate form:

$$\lambda_f : \mathrm{T} \operatorname{Coker} \widehat{f} \times \mathrm{T} \operatorname{Coker} \widehat{f} \to \mathbb{Q}/\mathbb{Z}.$$

It also follows from this sequence that $M^\sharp = M$ iff $G_f = 0$ iff f is unimodular.

Suppose next that (M, f, c) is a bilinear lattice equipped with a characteristic form $c \in M^*$. Denote by $c_{\mathbb{Q}} : V^* \to \mathbb{Q}$ the linear extension of c . We associate to (M, f, c) a quadratic function

$$\phi_{f,c} : G_f \to \mathbb{Q}/\mathbb{Z}$$

over L_f by putting:

$$\phi_{f,c}\left([x]\right) = \frac{1}{2}\left(f_{\mathbb{Q}}(x,x) - c_{\mathbb{Q}}(x)\right) \bmod \mathbb{Z}, \quad x \in M^\sharp.$$

Observe that the homogeneity defect $d_{\phi_{f,c}} \in G_f^* = \operatorname{Hom}\left(G_f, \mathbb{Q}/\mathbb{Z}\right)$ is given by:

$$d_{\phi_{f,c}}\left([x]\right) = -c_{\mathbb{Q}}(x) \bmod \mathbb{Z}, \quad x \in M^\sharp. \tag{2.5}$$

Definition 2.4. The triple (M, f, c) is said to be a *presentation* of the quadratic function $\phi_{f,c}$ on the torsion Abelian group G_f. The assignation $(M, f, c) \mapsto (G_f, \phi_{f,c})$ is called the *discriminant* construction.

Lemma 2.5. *The discriminant construction preserves orthogonal sums.*

Remark 2.6. The quadratic function $\phi_{f,c}$ factorizes *via* B_f to a (nondegenerate) quadratic function over λ_f if and only if c can be taken to be the image of a fractional Wu class.

In particular, when f is nondegenerate, this always happens (Lemma 2.1). In fact, $B_f : G_f \to$ Coker $\widehat{f}$ is then an isomorphism inducing a bijective correspondance:

$$\operatorname{Quad}\left(\lambda_f\right) \simeq \operatorname{Quad}\left(L_f\right).$$

This particular case coincides with the usual discriminant construction, as presented for example in [**De2**, §1.2.1].

2.2.3. *Properties of the discriminant construction.* Consider the bilinear map $M^* \times M^\sharp \to \mathbb{Q}/\mathbb{Z}$ defined by $(\alpha, x) \mapsto \alpha_\mathbb{Q}(x) \bmod \mathbb{Z}$. This map induces a bilinear pairing:

$$\langle -,-\rangle : \ \mathrm{Coker}\ \widehat{f} \times G_f \to \mathbb{Q}/\mathbb{Z}.$$

LEMMA 2.7. *The bilinear pairing $\langle -,-\rangle : \mathrm{Coker}\ \widehat{f} \times G_f \to \mathbb{Q}/\mathbb{Z}$ is left nondegenerate (respectively left nonsingular if and only if f is nondegenerate) and right nonsingular.*

For the proof of Lemma 2.7, two points have to be observed. First, by (2.3), any homomorphism $\mathrm{Ker}\ \widehat{f} \to \mathbb{Z}$ induces by tensoring with $\mathbb{Q}/\mathbb{Z}$ a homomorphism $\mathrm{Ker}\ \widehat{L_f} \to \mathbb{Q}/\mathbb{Z}$. We denote by

$$j_f : \left(\mathrm{Ker}\ \widehat{f}\right)^* \to \left(\mathrm{Ker}\ \widehat{L_f}\right)^*$$

the corresponding map. Second, the following lemma holds.

LEMMA 2.8. *The cokernel of the inclusion* $\mathrm{T}\ \mathrm{Coker}\ \widehat{f} \hookrightarrow \mathrm{Coker}\ \widehat{f}$ *is* $\left(\mathrm{Ker}\ \widehat{f}\right)^*$.

PROOF. Since $M/\mathrm{Ker}\ \widehat{f}$ is free, the inclusion $\mathrm{Ker}\ \widehat{f} \to M$ induces a surjective map $M^* \to \left(\mathrm{Ker}\ \widehat{f}\right)^*$ which sends $\widehat{f}(M)$ to 0. Hence we obtain a well-defined surjective map $\mathrm{Coker}\ \widehat{f} \to \left(\mathrm{Ker}\ \widehat{f}\right)^*$, the kernel of which, since $\left(\mathrm{Ker}\ \widehat{f}\right)^*$ is free of same rank as $\mathrm{Coker}\ \widehat{f}$, consists of all torsion elements. $\square$

PROOF OF LEMMA 2.7. We start with proving that the left adjoint map $\mathrm{Coker}\ \widehat{f} \to \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$ is injective. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{T}\ \mathrm{Coker}\ \widehat{f} \arrow[r] \arrow[d, "\widehat{\lambda_f}", "\simeq"'] & \mathrm{Coker}\ \widehat{f} \arrow[r] \arrow[d, ".\langle -,-\rangle"] & \left(\mathrm{Ker}\ \widehat{f}\right)^* \arrow[r] \arrow[d, "j_f"] & 0 \\
0 \arrow[r] & \left(\mathrm{T}\ \mathrm{Coker}\ \widehat{f}\right)^* \arrow[r] & \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \left(\mathrm{Ker}\ \widehat{L}_f\right)^* \arrow[r] & 0.
\end{tikzcd}$$

The first row is given by Lemma 2.8. The second row is obtained from the short exact sequence (2.4) by applying the exact functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$. As follows easily from definitions, that diagram is commutative. Then, by the "snake lemma", $\mathrm{Ker}\ .\langle -,-\rangle$ is isomorphic to $\mathrm{Ker}\ j_f$. The homomorphism j_f is essentially the map

$$- \otimes \mathbb{Q}/\mathbb{Z} : \mathrm{Hom}\left(\mathbb{Z}^k, \mathbb{Z}\right) \to \mathrm{Hom}\left((\mathbb{Q}/\mathbb{Z})^k, \mathbb{Q}/\mathbb{Z}\right)$$

(if k is the rank of $\mathrm{Ker}\widehat{f}$), which is essentially itself the k-fold product of the canonical map $\mathbb{Z} \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ defined by $1 \mapsto \mathrm{Id}_{\mathbb{Q}/\mathbb{Z}}$. The latter is injective, so is the map $.\langle -,-\rangle$.

We now prove that the bracket is left nonsingular if and only if f is nondegenerate. By the above diagram, the map $.\langle -,-\rangle$ is bijective if and only if $\mathrm{Coker}\ j_f = 0$. Thus, it suffices to show that the canonical map $\mathbb{Z} \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ defined by $1 \mapsto \mathrm{Id}_{\mathbb{Q}/\mathbb{Z}}$ is not surjective. Let A_p denote the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of all elements $m/p^k \pmod 1$, $m \in \mathbb{Z}, k \in \mathbb{N}$. The Chinese remainder theorem implies that $\mathbb{Q}/\mathbb{Z} \simeq \oplus_{p \in \mathcal{P}} A_p$, where $\mathcal{P}$ denotes the set of all prime numbers. Consider the map $\pi_p : A_p \to A_p$ which is multiplication by p. Then the (strong) product $\prod_p \pi_p \in \prod_p \mathrm{Hom}(A_p, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ does not come from $\mathbb{Z}$.

Consider now the right adjoint map $G_f \to \mathrm{Hom}\left(\mathrm{Coker}\ \widehat{f}, \mathbb{Q}/\mathbb{Z}\right)$. An element $[x] \in G_f$ lies in the kernel if and only if $\alpha_\mathbb{Q}(x) = 0 \bmod \mathbb{Z}$ for all $\alpha \in M^*$. Clearly, this implies that $x \in M$, thus $[x] = 0$.

We now prove that the right adjoint map is surjective. Consider the canonical isomorphism $V \to V^{**}, v \mapsto \langle -, v\rangle$. This map sends $M^\sharp$ onto $N = \{\langle -, w\rangle \in V^{**} : \langle \widehat{f}(M), w\rangle \subset \mathbb{Z}\}$ which is a subgroup of V^{**}. The group $N^0 = \{x \in \mathrm{Hom}(M^*, \mathbb{Q}) : \langle \widehat{f}(M), x\rangle \subset \mathbb{Z}\}$ embeds in N by linear extension over $\mathbb{Q}$: $M^* \to M^* \otimes \mathbb{Q} = V^*$. Any $x \in N^0$ induces a map $\mathrm{Coker}\ \widehat{f} \to \mathbb{Q}/\mathbb{Z}$, hence there is an induced homomorphism $N^0 \to \mathrm{Hom}\left(\mathrm{Coker}\ \widehat{f}, \mathbb{Q}/\mathbb{Z}\right)$, which is obviously surjective (since M^* is $\mathbb{Z}$-free). Hence any $\varphi \in \mathrm{Hom}\left(\mathrm{Coker}\ \widehat{f}, \mathbb{Q}/\mathbb{Z}\right)$ lifts to an element $x \in N^0 \subset N$ where $x = \langle -, v\rangle$ for some $v \in M^\sharp$. Thus $\varphi = \langle -, [v]\rangle$. $\square$

For any $c \in \mathrm{Char}(f)$ and any $u \in M$, we have $\phi_{f,c+2\hat{f}(u)} = \phi_{f,c}$; hence $\phi_{f,c}$ depends on c only mod $2\widehat{f}(M)$. Also, the Abelian group $M^*/\widehat{f}(M) = \mathrm{Coker}\ \widehat{f}$ acts freely and transitively on $\mathrm{Char}(f)/2\widehat{f}(M)$ by:

$$[\alpha] \cdot [c] = [c + 2\alpha] \in \mathrm{Char}(f)/2\widehat{f}(M),\ \alpha \in M^*,\ c \in \mathrm{Char}(f).$$

The next theorem is the main result of this section.

THEOREM 2.9. *The map $[c] \mapsto \phi_{f,c}$ induces an affine embedding*

$$\phi_f : \mathrm{Char}(f)/2\widehat{f}(M) \hookrightarrow \mathrm{Quad}(L_f)$$

over the opposite of the group monomorphism $.\langle -, -\rangle : \mathrm{Coker}\ \widehat{f} \hookrightarrow \mathrm{Hom}\,(G_f, \mathbb{Q}/\mathbb{Z})$. *Furthermore,*

$$\mathrm{Coker}\ \phi_f \simeq \mathrm{Coker}\ j_f = \frac{\left(\mathrm{Ker}\ \widehat{L_f}\right)^*}{j_f\left(\left(\mathrm{Ker}\ \widehat{f}\right)^*\right)}$$

and given $q \in \mathrm{Quad}(L_f)$, q belongs to $\mathrm{Im}\ \phi_f$ *if and only if* $q|_{\mathrm{Ker}\ \widehat{L_f}}$ *belongs to* $\mathrm{Im}\ j_f$.

COROLLAIRE 2.1. *The map ϕ_f is bijective if and only if f is nondegenerate.*

PROOF OF THEOREM 2.9. Let us prove that the map ϕ_f is indeed affine over the homomorphism stated above. Let $\alpha \in M^*$ and $x \in M^\sharp$: $\phi_{f,[\alpha]\cdot[c]}([x]) - \phi_{f,[c]}([x]) = \phi_{f,c+2\alpha}([x]) - \phi_{f,c}([x]) = -\alpha_{\mathbb{Q}}(x) \bmod \mathbb{Z}$. Hence:

$$\phi_{f,[\alpha]\cdot[c]} = \phi_{f,[c]} - \langle [\alpha], -\rangle.$$

The fact that ϕ_f is injective follows from Lemma 2.7. Next, $\mathrm{Coker}\ \phi_f = \mathrm{Coker}\ .\langle -, -\rangle$ is also determined from the commutative diagram drawn in the proof of Lemma 2.7. We have $\mathrm{Coker}\ .\langle -, -\rangle \simeq \mathrm{Coker}\ j_f$.

Finally we prove the last statement of the theorem. The homomorphism which arises as the restriction of $\phi_{f,c}$ to $\mathrm{Ker}\ \widehat{L_f} = \left(M + \mathrm{Ker}\ \widehat{f_{\mathbb{Q}}}\right)/M$ is given by:

$$\phi_{f,c}([x]) = -\frac{1}{2}c_{\mathbb{Q}}(x) \bmod \mathbb{Z},\quad x \in M + \mathrm{Ker}\ \widehat{f}_{\mathbb{Q}} \subset M^\sharp. \tag{2.6}$$

Hence $\phi_{f,c}|_{\mathrm{Ker}\ \widehat{L_f}} = j_f\left(-\frac{1}{2}c\right) \in \mathrm{Im}\ j_f$, where we use the fact that $c\left(\mathrm{Ker}\ \widehat{f}\right) \subset 2\mathbb{Z}$. Conversely, let $q \in \mathrm{Quad}(L_f)$ such that $q|_{\mathrm{Ker}\ \widehat{L_f}} \in \mathrm{Im}\ j_f$. Pick a $c \in \mathrm{Char}(f)$. Then the restriction map $(G_f)^* \to \left(\mathrm{Ker}\ \widehat{L_f}\right)^*$ sends $q - \phi_{f,c}$ to $\mathrm{Im}\ j_f$. Since $\mathrm{Coker}\ .\langle -, -\rangle \simeq \mathrm{Coker}\ j_f$, there is an element $[\alpha] \in \mathrm{Coker}\ \widehat{f}$ such that $q - \phi_{f,c} = \langle [\alpha], -\rangle$. Since the map ϕ_f is affine over $\mathrm{Coker}\ \widehat{f} \hookrightarrow \mathrm{Hom}\,(G_f, \mathbb{Q}/\mathbb{Z})$, it follows that $q = \phi_{f,[-\alpha]\cdot[c]}$. $\square$

2.3. The stable classification theorem. The goal here is to generalize a result due to Wall and Durfee ([**W2**, Corollary 1] and [**Du**, Corollary 4.2.(ii)]). We define a natural notion of stable equivalence on lattices equipped with a characteristic form. The resulting stable classification problem is shown to be equivalent to the classification of the quadratic functions induced by the discriminant construction (§2.2).

There is a natural notion of isomorphism among triples (M, f, c) defined by bilinear lattices with characteristic forms: we say that two triples (M, f, c) and (M', f', c') are isomorphic if there is an isomorphism $\psi : M \to M'$ such that $\psi^* f' = f$ and $\psi^* c' = c \bmod 2\widehat{f}(M)$. All such triples form a monoid for the orthogonal sum $\oplus$. Two triples (M, f, c) and (M', f', c') are said to be *stably equivalent* if they become isomorphic after stabilizations with some unimodular lattices, that is, there is an isomorphism (called a *stable equivalence*) between $(M, f, c) \oplus (U, g, s)$ and $(M', f', c') \oplus (U', g', s')$ for some unimodular lattices (U, g, s) and (U', g', s') equipped with characteristic forms s and s' respectively. Because, as we have seen in §2.2, unimodular lattices induce trivial discriminant bilinear forms and quadratic functions, a stable equivalence between two triples induce an isomorphism of the corresponding quadratic functions. We are interested in whether the converse holds and to which extent. In fact, a positive answer is provided in the case of nondegenerate lattices.

PROPOSITION 2.10. *Two nondegenerate symmetric bilinear lattices (M, f, c) and (M', f', c') equipped with characteristic forms are stably equivalent if and only if their associated quadratic functions $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ are isomorphic. In fact, any isomorphism ψ between their associated quadratic functions $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ lifts to a stable equivalence between (M, f, c) and (M', f', c').*

PROOF. This is proved in [LW, §1.7.3] which relies on [W2]. □

In the general case, we have to deal with the potential degeneracy of lattices. The first observation is that it is *not true* that any isomorphism between $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ will lift to a stable equivalence between (M, f, c) and (M', f', c'). In fact, the simplest counterexample is given by the degenerate bilinear lattice $(\mathbb{Z}, 0, 0)$ (with 0 as characteristic form). We have $G_0 = \mathbb{Q}/\mathbb{Z}$, $\phi_{0,0} = 0$. Trivially any automorphism of $\mathbb{Q}/\mathbb{Z}$ is an automorphism of the degenerate quadratic function $(\mathbb{Q}/\mathbb{Z}, 0)$. However, not every automorphism of $\mathbb{Q}/\mathbb{Z}$ lifts to an automorphism of $\mathbb{Q}$.
For each bilinear lattice (M, f, c), Lemma 2.7 gives an isomorphism $\langle -, - \rangle_{\bullet} : G_f \to \mathrm{Hom}\left(\mathrm{Coker}\ \widehat{f}, \mathbb{Q}/\mathbb{Z}\right)$. Therefore any isomorphism $\psi : \mathrm{Coker}\ \widehat{f} \to \mathrm{Coker}\ \widehat{f'}$ induces an isomorphism $\psi^\sharp : G_{f'} \to G_f$. The corresponding map:

$$\mathrm{Iso}\left(\mathrm{Coker}\ \widehat{f}, \mathrm{Coker}\ \widehat{f'}\right) \to \mathrm{Iso}\left(G_{f'}, G_f\right)$$

is injective (but not surjective, unless f, f' are nondegenerate: see Lemma 2.13 below). We now state the main theorem of this section.

THEOREM 2.11. *Let (M, f, c) and (M', f', c') be some bilinear lattices with characteristic forms. Then, they are stably equivalent if and only if their associated quadratic functions $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ are isomorphic via an element $\psi^\sharp \in \mathrm{Im}\left(\mathrm{Iso}\left(\mathrm{Coker}\ \widehat{f}, \mathrm{Coker}\ \widehat{f'}\right) \to \mathrm{Iso}\left(G_{f'}, G_f\right)\right)$. Furthermore, any such isomorphism between $(G_{f'}, \phi_{f',c'})$ and $(G_f, \phi_{f,c})$ lifts to a stable equivalence between (M, f, c) and (M', f', c').*

In the sequel, (M,f,c) and (M',f',c') denote bilinear lattices with characteristic forms. We also put: $N_f = \mathrm{Ker}\ \widehat{L_f}$ and $N_{f'} = \mathrm{Ker}\ \widehat{L_{f'}}$.

LEMMA 2.12. *Any isomorphism* $\psi : \mathrm{Coker}\ \widehat{f} \to \mathrm{Coker}\ \widehat{f'}$ *induces an isomorphism* $\psi^\sharp : \mathrm{Ker}\ \widehat{f'} \to \mathrm{Ker}\ \widehat{f}$. *Furthermore, if* $\psi([c]) = [c']$ *then* $\psi^\sharp$ *is an isomorphism from the triple* $\left(\mathrm{Ker}\ \widehat{f'}, 0, c'|_{\mathrm{Ker}\ \widehat{f'}}\right)$ *onto the triple* $\left(\mathrm{Ker}\ \widehat{f}, 0, c|_{\mathrm{Ker}\ \widehat{f}}\right)$.

PROOF. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{T\ Coker}\ \widehat{f} \arrow[r] \arrow[d, "\psi|_{\mathrm{TCoker}\ \widehat{f}}"', "\simeq"] & \mathrm{Coker}\ \widehat{f} \arrow[r] \arrow[d, "\simeq"', "\psi"] & \left(\mathrm{Ker}\ \widehat{f}\right)^* \arrow[r] \arrow[d, "{[\psi]}"] & 0 \\
0 \arrow[r] & \mathrm{T\ Coker}\ \widehat{f'} \arrow[r] & \mathrm{Coker}\ \widehat{f'} \arrow[r] & \left(\mathrm{Ker}\ \widehat{f'}\right)^* \arrow[r] & 0
\end{tikzcd}$$

where the two exact rows are given by Lemma 2.8 and where $[\psi]$ is induced by ψ and is an isomorphism. Applying $\mathrm{Hom}(-,\mathbb{Z})$ to $[\psi]$ yields an isomorphism $[\psi]^* : \left(\mathrm{Ker}\ \widehat{f'}\right)^{**} = \mathrm{Ker}\ \widehat{f'} \to \left(\mathrm{Ker}\ \widehat{f}\right)^{**} = \mathrm{Ker}\ \widehat{f}$ which is taken to be $\psi^\sharp$. For the second statement, note that $\left(\psi^\sharp\right)^* : \left(\mathrm{Ker}\ \widehat{f}\right)^* \to \left(\mathrm{Ker}\ \widehat{f'}\right)^*$ is $[\psi]$. Then, $c|_{\mathrm{Ker}\ \widehat{f}} \circ \psi^\sharp = [\psi]\left(c|_{\mathrm{Ker}\ \widehat{f}}\right) = c'|_{\mathrm{Ker}\ \widehat{f'}}$ since $\psi\left([c]\right) = [c']$. □

LEMMA 2.13. *Let* $\Psi \in \mathrm{Iso}(G_{f'}, G_f)$. *Then, the following assertions are equivalent:*

(1) *there exists* $\psi \in \mathrm{Iso}\left(\mathrm{Coker}\ \widehat{f}, \mathrm{Coker}\ \widehat{f'}\right)$ *such that* $\Psi = \psi^\sharp$;
(2) $\Psi\left(N_{f'}\right) = N_f$ *and the map* $\Psi|_{N_{f'}} : N_{f'} \to N_f$ *lifts to an isomorphism* $\mathrm{Ker}\ \widehat{f'} \to \mathrm{Ker}\ \widehat{f}$.

PROOF. (1) $\Longrightarrow$ (2): Lemma 2.12 gives an isomorphism $\psi^\sharp : \mathrm{Ker}\ \widehat{f'} \to \mathrm{Ker}\ \widehat{f}$ and $\Psi = \psi^\sharp$ is given by Lemma 2.7. It follows from the definitions that $\Psi|_{N_{f'}} : N_{f'} \to G_f$ coincides with $\left(\psi^\sharp\right) \otimes \mathbb{Q}/\mathbb{Z} : N_{f'} = \mathrm{Ker}\ \widehat{f'} \otimes \mathbb{Q}/\mathbb{Z} \to \mathrm{Ker}\ \widehat{f} \otimes \mathbb{Q}/\mathbb{Z} = N_f \subset G_f$.

(2) $\Longrightarrow$ (1): A homomorphism $y : G_{f'} \to \mathbb{Q}/\mathbb{Z}$ belongs to the image of $\widehat{\langle -,- \rangle} : \mathrm{Coker}\ \widehat{f'} \to \mathrm{Hom}\left(G_{f'}, \mathbb{Q}/\mathbb{Z}\right)$ if and only if $y|_{N_{f'}} : N_{f'} \to \mathbb{Q}/\mathbb{Z}$ lifts to a homomorphism $\mathrm{Ker}\ \widehat{f'} \to \mathbb{Z}$. This follows from the commutative diagram in the proof of Lemma 2.7 (since $\mathrm{Coker}\ \widehat{\langle -,- \rangle} \simeq \mathrm{Coker}\ j_{f'}$). Thus, by hypothesis, for any $x \in \mathrm{Coker}\ \widehat{f}$, there exists a unique $\psi(x) \in \mathrm{Coker}\ \widehat{f'}$ satisfying:

$$\forall y' \in G_{f'}, \quad \langle \psi(x), y' \rangle = \langle x, \Psi(y') \rangle \in \mathbb{Q}/\mathbb{Z}.$$

It also follows that the assignation $\psi : x \mapsto \psi(x)$ is additive, and that ψ is in fact an isomorphism (since Ψ is bijective). By construction, ψ is such that $\Psi = \psi^\sharp$. □

PROOF OF THEOREM 2.11. The nondegenerate case is treated by Proposition 2.10. Consider now the general case. Let s be a stable equivalence between lattices, say an isomorphism of symmetric bilinear lattices $s : (M',f',c') \oplus (U',g',u') \to (M,f,c) \oplus (U,g,u)$ where (U',g',u') and (U,g,u) are unimodular lattices. The map s induces via the discriminant construction an isomorphism $\Psi : G_{f'} \to G_f$ between $\phi_{f',c'}$ and $\phi_{f,c}$ since unimodular lattices induce trivial quadratic functions. It also induces in the obvious way an isomorphism $\psi : \mathrm{Coker}\ \widehat{f} \to \mathrm{Coker}\ \widehat{f'}$, and it is easily verified that $\psi^\sharp = \Psi$.

Conversely, suppose given an isomorphism Ψ between $(G_{f'}, \phi_{f',c'})$ and $(G_f, \phi_{f,c})$ which is induced by an isomorphism $\psi : \mathrm{Coker}\ \widehat{f} \to \mathrm{Coker}\ \widehat{f'}$. Then, the homogeneity defects are preserved by Ψ: $d_{\phi_{f',c'}} = d_{\phi_{f,c}} \circ \Psi$. Since $\Psi = \psi^\sharp$, it follows from (2.5) that $\psi([c]) = [c']$. Thus Lemma 2.12 implies that $\left(\mathrm{Ker}\ \widehat{f'}, 0, c'|_{\mathrm{Ker}\ \widehat{f'}}\right)$ and $\left(\mathrm{Ker}\ \widehat{f}, 0, c|_{\mathrm{Ker}\ \widehat{f}}\right)$ are isomorphic *via* $\psi^\sharp : \mathrm{Ker}\ \widehat{f'} \to \mathrm{Ker}\ \widehat{f}$.

Let $(\bar{M}, \bar{f})$ be the nondegenerate lattice induced by f (see (2.1)), and pick a section s of the canonical projection $p : M \to \bar{M}$. Lemma 2.2 yields a characteristic form $\bar{c} = s^*|_{\mathrm{Char}(f)}(c) \in \mathrm{Char}(\bar{f})$. The section s of p induces a map $G_{\bar{f}} \to G_f$ still denoted by s; this map is such that $\phi_{\bar{f},\bar{c}} = \phi_{f,c} \circ s$.
Let also $s' : \bar{M}' \to M'$ be a section of the canonical projection $p' : M' \to \bar{M}'$ such that the inducep map $s' : G_{\bar{f}'} \to G_{f'}$ verifies:

$$\begin{tikzcd}
G_{f'} \arrow[d, "\Psi"'] & G_{\bar{f}'} \arrow[l, "s'"'] \arrow[r, "B_{\bar{f}'}", "\simeq"'] & \mathrm{T}\ \mathrm{Coker}\ \widehat{\bar{f}'} & \mathrm{T}\ \mathrm{Coker}\ \widehat{f'} \arrow[l, "\simeq"'] \\
G_f & G_{\bar{f}} \arrow[l, "s"] \arrow[r, "\simeq", "B_{\bar{f}}"'] & \mathrm{T}\ \mathrm{Coker}\ \widehat{\bar{f}} & \mathrm{T}\ \mathrm{Coker}\ \widehat{f}. \arrow[l, "\simeq"'] \arrow[u, "\psi|"]
\end{tikzcd}$$

Set $\bar{c'} = s'^*|_{\mathrm{Char}(f')}(c') \in \mathrm{Char}\left(\bar{f}'\right)$; then, $\phi_{\bar{f}',\bar{c}'} = \phi_{f',c'} \circ s'$. We deduce that: $\phi_{\bar{f}',\bar{c}'} \circ \psi| = \phi_{\bar{f},\bar{c}}$. Since these quadratic functions are nondegenerate, Proposition 2.10 applies: the isomorphism $\psi|$ lifts to a stable equivalence between the lattices $(\bar{M}, \bar{f}, \bar{c})$ and $(\bar{M}', \bar{f}', \bar{c}')$.

Hence finally, there is a stable equivalence between:

$$(M, f, c) \simeq (\bar{M}, \bar{f}, \bar{c}) \oplus \left(\mathrm{Ker}\ \widehat{f}, 0, c|_{\mathrm{Ker}\ \widehat{f}}\right) \quad \text{and:}$$

$$(M', f', c') \simeq (\bar{M}', \bar{f}', \bar{c}') \oplus \left(\mathrm{Ker}\ \widehat{f'}, 0, c'|_{\mathrm{Ker}\ \widehat{f'}}\right),$$

where the isomorphisms are respectively induced by the sections s and s'. By construction, the isomorphism $G_{f'} \to G_f$ induced by this stable equivalence is exactly Ψ. $\square$

Consider two bilinear forms, denoted ± 1, defined on $\mathbb{Z}$ by $(1,1) \mapsto \pm 1$, and both equipped with the Wu class $1 \in \mathbb{Z}$. For $n \in \mathbb{N}$, we denote by $n(\mathbb{Z}, \pm 1, 1)$ the n-fold orthogonal sum of $(\mathbb{Z}, \pm 1, 1)$.

COROLLARY 2.14. *Let (M, f, c) and (M', f', c') be some bilinear lattices equipped with characteristic forms. Then, the quadratic functions $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ are isomorphic if and only if there exists $n, n' \in \mathbb{N}$ such that $(M, f, c) \oplus n(\mathbb{Z}, \pm 1, 1) \simeq (M', f', c') \oplus n'(\mathbb{Z}, \pm 1, 1)$.*

PROOF. One direction is obvious. For the converse, we apply Theorem 2.11. We obtain unimodular lattices (U, g) and (U', g'), equipped with characteristic forms s and s' respectively, such that there is an isomorphism ψ sending $(M, f, c) \oplus (U, g, s)$ onto $(M', f', c') \oplus (U', g', s')$. By stabilizing if necessary with $(\mathbb{Z}, 1)$ and $(\mathbb{Z}, -1)$, we may assume that, as a symmetric bilinear pairing, (U, g) is indefinite[1] and not even[2]. Then a theorem asserts that (U, g) is isomorphic to an orthogonal sum of copies of $(\mathbb{Z}, 1)$ and $(\mathbb{Z}, -1)$ [**Se**]. It follows that (U, g, s) is isomorphic to a sum of copies of $(\mathbb{Z}, 1, 1)$ and $(\mathbb{Z}, -1, 1)$. (Here we use the fact that any odd integer $a \in \mathbb{Z}$ induces a characteristic form for $(\mathbb{Z}, \pm 1)$ and any such triple $(\mathbb{Z}, \pm 1, a)$ is isomorphic to $(\mathbb{Z}, \pm 1, 1)$.) A similar observation holds for (U', g', s'). We conclude

[1] i.e. g takes both positive and negative values.

[2] even means that $g(u, u) \in 2\mathbb{Z}$ for all $u \in U$.

that there is a stable equivalence between (M, f, c) and (M', f', c') involving only a stabilization with copies of $(\mathbb{Z}, 1, 1)$ and $(\mathbb{Z}, -1, 1)$, which is the desired result. $\square$

REMARK 2.15. This is a generalization of [**De2**, Lemma 2.1(b)] (whose proof contains a misprint).

2.4. A complete system of invariants. We present a complete system of invariants for nondegenerate quadratic functions on finite Abelian groups.
Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate quadratic function on a finite Abelian group G. Its *Gauss sum* is:

$$\gamma(q) = |G|^{-\frac{1}{2}} \sum_{x \in G} e^{2i\pi q(x)} \in \mathbf{S}^1.$$

An important observation is the relation:

$$\gamma(\alpha \cdot q) = e^{-2i\pi q(\alpha)} \gamma(q), \ \alpha \in G. \tag{2.7}$$

Recall that b_q denotes the associated bilinear pairing of q and $d_q \in G^*$ is its homogeneity defect. The following result is the main result of this section.

THEOREM 2.16. *Two nondegenerate quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ on finite Abelian groups are isomorphic if and only if there is an isomorphism $\psi : G \to G'$ such that $\psi^* b_{q'} = b_q$, $\psi^* d_{q'} = d_q$ and $\gamma(q') = \gamma(q)$.*

COROLLARY 2.17. *Two nondegenerate homogeneous quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ on finite Abelian groups are isomorphic if and only if there is an isomorphism $\psi : G \to G'$ such that $\psi^* b_{q'} = b_q$ and $\gamma(q') = \gamma(q)$.*

That two isomorphic quadratic functions have same Gauss sums and isomorphic associated bilinear pairings and associated homomorphisms is obvious. The difficult part lies in the converse. The key step is the following lemma.

LEMMA 2.18 (Fundamental Lemma). *Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate quadratic function on a finite Abelian group. If $\alpha \in G$ is an element of order 2 such that $q(\alpha) = 0$, then $\alpha \cdot q \sim q$.*

PROOF. Define a homomorphism $\eta : G \to \mathbb{Z}_2$ by

$$\forall x \in G, \quad \frac{1}{2} \cdot \eta(x) = b_q(\alpha, x) \in \mathbb{Q}/\mathbb{Z}$$

where $\frac{1}{2}\cdot$ denotes the usual inclusion of $\mathbb{Z}_2$ into $\mathbb{Q}/\mathbb{Z}$. In the sequel, for any $c \in \mathbb{Z}_2$, we put

$$G \ni c \cdot \alpha = \begin{cases} \alpha & \text{if } c = 1, \\ 0 & \text{if } c = 0. \end{cases}$$

Observe that $(c_1 + c_2) \cdot \alpha = c_1 \cdot \alpha + c_2 \cdot \alpha$ for any $c_1, c_2 \in \mathbb{Z}_2$ since $2\alpha = 0$.
Let now $\psi : G \to G$ be the map defined by $\psi(x) = x + \eta(x) \cdot \alpha$ for all $x \in G$. As η is a homomorphism, ψ is an endomorphism of G. Furthermore, ψ is an involution since for any $x \in G$, we have

$$\begin{aligned} \psi^2(x) &= \psi\left(x + \eta(x) \cdot \alpha\right) \\ &= x + \eta(x) \cdot \alpha + (\eta(x) + \eta(\eta(x) \cdot \alpha)) \cdot \alpha \\ &= x + 0 + (\eta(x)\eta(\alpha)) \cdot \alpha \\ &= x. \end{aligned}$$

Last identity follows from the fact that $\frac{1}{2} \cdot \eta(\alpha) = b_q(\alpha, \alpha) = q(2\alpha) - 2q(\alpha) = 0$. Lastly, for any $x \in G$, we have

$$\begin{aligned} q(\psi(x)) &= q(x + \eta(x) \cdot \alpha) \\ &= q(x) + q(\eta(x) \cdot \alpha) + b_q(x, \eta(x) \cdot \alpha) \\ &= q(x) + 0 + \frac{1}{2} \cdot \eta(x) \\ &= (\alpha \cdot q)(x). \end{aligned}$$

Therefore, q is isomorphic to $\alpha \cdot q$ *via* the involution ψ. □

PROOF OF TH. 2.16. With no loss of generality, we may assume that $G = G'$, $b_q = b_{q'}$, $d_q = d_{q'}$ and $\gamma(q) = \gamma(q')$ and then show that $q \sim q'$. The equality $b_q = b_{q'}$ implies (cf. §2.1) that $q' = \alpha \cdot q$ for some $\alpha \in G$. The equality $d_q = d_{q'}$ implies $2\alpha = 0$. The equality $\gamma(q) = \gamma(q')$, together with (2.7) imply that $q(\alpha) = 0$. Hence Lemma 2.18 applies. □

For further use, we need a version of Theorem 2.16 which allows for some degeneracy.

LEMMA 2.19. *Consider the following commutative diagram of extensions of Abelian groups:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\psi|_A", "\simeq"'] & G \arrow[r, "p"] \arrow[d, "\psi", "\simeq"'] & B \arrow[r] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & G' \arrow[r, "p'"] & B' \arrow[r] & 0. \end{tikzcd}$$

where the extension of A by B is split. Then the extension of A' is also split.

PROOF. Let $[\psi] : B \to B'$ be the isomorphism induced by ψ. Let also $s : B \to G$ be a section of p. Then $s' = \psi \circ s \circ [\psi]^{-1}$ is a section of p'. □

DEFINITION 2.20. Given a diagram as in Lemma 2.19, two sections s and s' of p and p' respectively are said *ψ-compatible*, or simply *compatible*, if they are related as in the proof of Lemma 2.19:

$$\begin{tikzcd} G \arrow[r, "p"'] \arrow[d, "\psi"'] & B \arrow[l, bend right, "s"'] \arrow[d, "{[\psi]}"] \\ G' \arrow[r, "p'"] & B' \arrow[l, bend left, "s'"] \end{tikzcd}$$

Let now $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on a Abelian group. Then, G can be regarded as the extension of $A = \text{Ker } \widehat{b_q}$ by $B = G/\text{Ker } \widehat{b_q}$. We shall say that (G, q) meets the *finiteness condition* if the following two conditions hold:

- $G/\text{Ker } \widehat{b_q}$ is finite;
- the extension G of $\text{Ker } \widehat{b_q}$ by $G/\text{Ker } \widehat{b_q}$ is split.

EXAMPLE 2.21. Let (M, f, c) be a bilinear lattice equipped with a characteristic form c. Then, the quadratic function $(G_f, \phi_{f,c})$ meets the finiteness condition since the short exact sequence (2.4) is split (a section of its is induced by any section of the canonical projection $M \to \bar{M} = M/\text{Ker } \widehat{f}$).

In the sequel, we denote by $r_q : \text{Ker } \widehat{b_q} \to \mathbb{Q}/\mathbb{Z}$ the homomorphism $q|_{\text{Ker } \widehat{b_q}}$.

COROLLARY 2.22. *Let $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ be two quadratic functions on Abelian groups satisfying the finiteness condition. Then, they are isomorphic if and only if there is an isomorphism $\psi : G \to G'$ such that $\psi^* b_{q'} = b_q$, $\psi^* d_{q'} = d_q$, $\psi^* r_{q'} = r_q$ and $\gamma(q' \circ s') = \gamma(q \circ s)$ for some ψ-compatible sections s and s'.*

PROOF. Consider $q_1 = q \circ s$ and $q_1' = q' \circ s'$: they are nondegenerate quadratic functions on finite Abelian groups. Let ψ_1 be the isomorphism $[\psi] : G/\text{Ker } \widehat{b_q} \to G'/\text{Ker } \widehat{b_{q'}}$ induced by ψ. We then have: $\psi_1^* b_{q_1'} = b_{q_1}$, $\psi_1^* d_{q_1'} = d_{q_1}$ and $\gamma(q_1) = \gamma(q_1')$. Then, by Theorem 2.16, q_1 and q_1' are isomorphic. We then deduce that $(G, q) \simeq \left(G/\text{Ker } \widehat{b_q}, q_1\right) \oplus \left(\text{Ker } \widehat{b_q}, r_q\right)$ is isomorphic to $(G', q') \simeq \left(G'/\text{Ker } \widehat{b_{q'}}, q_1'\right) \oplus \left(\text{Ker } \widehat{b_{q'}}, r_{q'}\right)$. □

3. 3-manifolds with complex spin structures

We now give a treatment of Spin^c-manifolds. We begin by some generalities and then specialize our exposition to the dimension 3.

3.1. Complex spin structures. The homotopy-theoritical exposition of Spin^c-structures we give here, follows *mutatis mutandis* from an analogous description of Spin-structures given by Blanchet and Masbaum in [**BM**].

We first fix a few conventions. Vector bundles will be stabilized from left. If G is a group, $\omega_G : EG \to BG$ will denote the universal principal G-bundle. The map induced by a bundle morphism F on the base spaces is denoted by the corresponding lower case letter f. All manifolds are assumed to be compact, smooth and oriented. The real interval $[0, 1]$ will be denoted by I. Unless otherwise specified, all (co)homology groups are assumed to be with integer coefficients.

3.1.1. *The* Spin^c *group.* The Spin group is the 2-fold covering of the special orthogonal group SO:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \text{Spin} \xrightarrow{\pi} SO \to 1.$$

The group Spin^c is defined by:

$$\text{Spin}^c = \frac{\text{Spin} \times U(1)}{\mathbb{Z}/2\mathbb{Z}},$$

where $\mathbb{Z}/2\mathbb{Z}$ is generated by $[(-1, -1)]$, hence this short exact sequence of groups:

$$1 \longrightarrow U(1) \xrightarrow{j} \text{Spin}^c \xrightarrow{\rho} SO \longrightarrow 1,$$

where j sends z to $[(1, z)]$, and where ρ sends $[(x, z)]$ to $\pi(x)$. From this, we obtain the following fibration for classifying spaces:

$$BU(1) \rightarrowtail^{Bj} B\text{Spin}^c \overset{B\rho}{\twoheadrightarrow} BSO.$$

Denote by γ_{SO} the universal stable oriented vector bundle over BSO, and by γ_{Spin^c} the pull-back of γ_{SO} by $B\rho$. Analogously, for any $n \geq 1$, starting from $SO(n)$ we define the groups $\text{Spin}(n)$ and $\text{Spin}^c(n)$.

3.1.2. *Rigid* Spin^c*-structures.* Let M be a n-manifold (assumed to be compact smooth and oriented by convention). We denote by T_M (resp. τ_M) its oriented (resp. and stable) tangent bundle.

DEFINITION 3.1. A *rigid* Spin^c*-structure* on M is a vector bundle morphism $G : \tau_M \to \gamma_{\text{Spin}^c}$ which is orientation-preserving on each fiber. A Spin^c*-structure* (or *complex spin structure*) on M is a homotopy class of rigid Spin^c-structures on M. We denote by $\text{Spin}^c_r(M)$ the set of rigid Spin^c-structures on M, and by $\text{Spin}^c(M)$ the set of its Spin^c-structures.

In the sequel, the letter β will stand for a Bockstein homomorphism associated to the following short exact sequence of coefficients:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

We now recall a well-known fact concerning existence and parametrization of Spin^c-structures.

PROPOSITION 3.2. *The manifold M can be given a* Spin^c*-structure if and only if the cohomology class:*

$$w(M) := \beta w_2(M) \in H^3(M)$$

vanishes, where $w_2(M)$ is the second Stiefel-Whitney class of M. If so, $\mathrm{Spin}^c(M)$ *is an affine space over the Abelian group $H^2(M)$.*

We use the multiplicative notation to denote this affine action. Thus, when $w(M) = 0$ and given $\sigma, \sigma' \in \mathrm{Spin}^c(M)$, the unique $h \in H^2(M)$ such that $\sigma' = h \cdot \sigma$ is denoted by σ'/σ.

PROOF OF PROPOSITION 3.2. Suppose that $G : \tau_M \to \gamma_{\mathrm{Spin}^c}$ is a rigid Spin^c-structure on M. By composing G with the canonical morphism $\gamma_{\mathrm{Spin}^c} \to \gamma_{SO}$, we obtain a morphism $F : \tau_M \to \gamma_{SO}$. At the level of base spaces, $f : M \to BSO$ is a classifying map for τ_M and $g : M \to B\mathrm{Spin}^c$ is then a lift of f by $B\rho$. By construction of γ_{Spin^c}, the rigid Spin^c-structure G on M *is equivalent* to the data (F, g) where:

(1) $F : \tau_M \to \gamma_{SO}$ is a morphism of oriented vector bundles,
(2) g is a lift of f by $B\rho$.

Thus, giving us a Spin^c-structure on M is equivalent to giving us a homotopy class of such pairs (F, g). Suppose now that *the bundle map F is fixed.* The space of bundle maps:

$$\mathrm{Map}_{SO}(\tau_M, \gamma_{SO})$$

is arc connected and contractible (see for instance [**Hu**, Ch. 7, Prop. 3.3] and [**Hu**, Ch. 7, Th. 3.4]). It can then be deduced from these two facts that giving us a Spin^c-structure on M is the same as giving us a lift g of f by $B\rho$, up to lift homotopy. This useful observation allows us to apply usual obstruction theory to the fibration $B\rho : B\mathrm{Spin}^c \to BSO$. This is a principal fibration with fiber $BU(1) \simeq BK(\mathbb{Z}, 1) \simeq K(\mathbb{Z}, 2)$, and with characteristic class $w := \beta w_2 \in H^3(BSO)$. The proposition then follows. $\square$

Let us now explain why our definition of Spin^c-structures agrees with the usual one.

LEMMA 3.3. *Suppose that M comes equipped with a Riemannian metric and denote by F_M the bundle of its oriented orthonormal frames. Then a* Spin^c*-structure on M is equivalent to an isomorphism class of pairs (η, H), where η is a principal* $\mathrm{Spin}^c(n)$*-bundle and where $H : \eta/U(1) \to F_M$ is an isomorphism of principal $SO(n)$-bundles.*

PROOF. Let (η, H) be such a pair: let us prove that it determines a rigid Spin^c-structure G on M by showing that it determines an equivalent data (F, g), as described in the proof of Prop. 3.2. Pick a morphism of $\mathrm{Spin}^c(n)$-bundles $\eta \to \omega_{\mathrm{Spin}^c(n)}$ which, when composed with the canonical map $\omega_{\mathrm{Spin}^c(n)} \to \omega_{\mathrm{Spin}^c}$, gives a certain $\tilde{G} : \eta \to \omega_{\mathrm{Spin}^c}$. This $\tilde{G}$ induces a morphism $\eta/U(1) \to \omega_{SO}$; by composing it with H^{-1}, we obtain $\tilde{F} : F_M \to \omega_{SO}$. The map $\tilde{F}$ then induces a morphism $F : \tau_M \to \gamma_{SO}$. We put $g := \tilde{g}$, induced by $\tilde{G}$ at the level of base spaces. Then, the assignation $(\eta, H) \mapsto (F, g) \equiv G$ induces the announced correspondance between the two definitions of Spin^c-structures. $\square$

3.1.3. *Relative* Spin^c*-structures.* Suppose in this paragraph that M is a n-manifold with non-empty boundary. We orient ∂M with the "outward normal vector first" rule; then since vector bundles are stabilized from left, the bundle $\tau_{\partial M}$ can be identified with $\tau_M|_{\partial M}$, hence a restriction map $\mathrm{Spin}^c_r(M) \to Spin^c_r(\partial M)$.

Definition 3.4. For a fixed $s \in \mathrm{Spin}^c_r(\partial M)$, a Spin^c*-structure on M relative to s* is a homotopy class rel ∂M of rigid Spin^c-structures on M which extend s. We denote by $\mathrm{Spin}^c(M,s)$ the set of such structures.

Obstruction theory can again be applied to obtain the following relative analog of Proposition 3.2:

Proposition 3.5. *Let $s \in \mathrm{Spin}^c_r(\partial M)$ be fixed. Then, s can be extended to M if and only if a certain cohomology class:*

$$w(M,s) \in H^3(M,\partial M)$$

vanishes. If so, $\mathrm{Spin}^c(M,s)$ is an affine $H^2(M,\partial M)$-space. Furthermore, the restriction of $w(M,s)$ to M is $w(M) = \beta w_2(M) \in H^3(M)$.

3.1.4. *Orientation reversal.* We denote by $-M$ the manifold obtained from M by reversing its orientation. There exists a map $\mathrm{Spin}^c_r(M) \to \mathrm{Spin}^c_r(-M)$ inducing a $H^2(M)$-equivariant map:

$$\mathrm{Spin}^c(M) \xrightarrow{\ -\ } \mathrm{Spin}^c(-M),$$

It is defined by composing rigid structures with the orthogonal reflection $\tau_{-M} \to \tau_M$ in the direction of the first stabilization.

Suppose that $\partial M \neq \varnothing$ and let $s \in \mathrm{Spin}^c_r(\partial M)$. Pick a nonsingular section v of τ_M extending the outward normal vector field on ∂M (v is well-defined up to relative homotopy). By composing rigid structures with the orthogonal reflection $\tau_{-M} \to \tau_M$ in the direction of v, we obtain a map:

$$\mathrm{Spin}^c(M,s) \xrightarrow{\ -\ } \mathrm{Spin}^c(-M,-s),$$

which is $H^2(M,\partial M)$-equivariant. It does not depend on v.

3.1.5. *From* Spin *to* Spin^c. Replacing Spin by Spin^c in what has been previously done, we can analogously define *rigid Spin-structures* and recover the usual Spin-structures: see [**BM**][3] for details. In particular, the bundle γ_{Spin} is defined to be the pull-back of γ_{SO} by $B\pi$. Consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{Spin} \arrow[r, "\beta"] \arrow[dr, "\pi"'] & \mathrm{Spin}^c \arrow[d, "\rho"] \\
& SO
\end{tikzcd}$$

where the morphism β is defined by $\beta(x) = [(x,1)]$. We then have $B\pi = B\rho \circ B\beta$ at the level of classifying spaces, hence a bundle map $\gamma_{\mathrm{Spin}} \to \gamma_{\mathrm{Spin}^c}$, defined in the obvious way.

So, if M is a n-manifold, there exists a map $\mathrm{Spin}_r(M) \to \mathrm{Spin}^c_r(M)$. It induces a canonical map:

$$\mathrm{Spin}(M) \xrightarrow{\ \beta\ } \mathrm{Spin}^c(M),$$

which is affine over the Bockstein homomorphism $\beta : H^1(M;\mathbb{Z}_2) \to H^2(M)$.

If now M has non-empty boundary and if $s \in \mathrm{Spin}_r(\partial M)$, there is also a map:

$$\mathrm{Spin}(M,s) \xrightarrow{\ \beta\ } \mathrm{Spin}^c\left(M,\beta(s)\right),$$

[3] Rigid Spin-structures are called "w_2-structures" in [**BM**]. Following this terminology, our rigid Spin^c-structures might have been called "βw_2-structures".

which is affine over the Bockstein homomorphism $\beta : H^1(M,\partial M;\mathbb{Z}_2) \to H^2(M,\partial M)$.

3.1.6. *Chern classes.* Let M be a n-manifold and let $\alpha \in \mathrm{Spin}^c(M)$. Because of the group homomorphism $\mathrm{Spin}^c(n) \to U(1)$ defined by $[(x,y)] \mapsto y^2$ and since α induces an isomorphism class of principal Spin^c(n)-bundles over M (by Lemma 3.3), α then determines an isomorphism class of principal $U(1)$-bundles. We denote by $c(\alpha)$ its first Chern class.

DEFINITION 3.6. The class $c(\alpha) \in H^2(M)$ is called the *Chern class of the* Spin^c*-structure* α. When $c(\alpha) \in \mathrm{T}H^2(M)$, the torsion subgroup of $H^2(M)$, the Spin^c-structure α is said to be *torsion.*

REMARK 3.7. The Chern class map $c : \mathrm{Spin}^c(M) \to H^2(M)$, is affine over the square[4] map $H^2(M) \to H^2(M)$ defined by: $x \mapsto x^2$. It follows that $\alpha \in \mathrm{Spin}^c(M)$ comes from $\mathrm{Spin}(M)$ by the map β defined at §3.1.5 if and only if $c(\alpha) = 0$.

3.2. Spin^c**-structures in dimension** 3. We now give a more specific treatment of Spin^c-structures in dimension 3. Manifolds are still supposed to be smooth compact and oriented. Note that any 3-manifold M can be given a Spin^c-structure since $w_2(M) = 0$.

3.2.1. *About* $SO(3)$*,* $\mathrm{Spin}(3)$ *and* $\mathrm{Spin}^c(3)$. Let $\mathbb{H}$ be the quaternion algebra:

$$\mathbb{H} = \{q = a + b\cdot i + c\cdot j + d\cdot k : a,b,c,d \in \mathbb{R}\}$$

and let $\mathbb{R}^3$ be the subspace of pure quaternions: $\mathbb{R}^3 = \mathbb{R}\cdot i + \mathbb{R}\cdot j + \mathbb{R}\cdot k$. The multiplicative group of unitary quaternions $\mathbf{S}^3 = \{q = a + b\cdot i + c\cdot j + d\cdot k : a^2+b^2+c^2+d^2 = 1\}$ can be identified with $SU(2)$ by:

$$q = a + b.i + c.j + d.k \mapsto \begin{pmatrix} a+b.i & -d+c.i \\ d+c.i & a-b.i \end{pmatrix}$$

We then define an homomorphism $\pi : SU(2) \to SO(3)$, sending a unitary quaternion q to the transformation:

$$\begin{array}{rccc} \pi(q) : & \mathbb{R}^3 & \to & \mathbb{R}^3 \\ & h & \mapsto & qhq^{-1}. \end{array}$$

The map π is a connected two-fold covering. Hence we can put $\mathrm{Spin}(3) = SU(2)$. Because of the isomorphism:

$$\frac{SU(2)\times U(1)}{\mathbb{Z}_2} \xrightarrow{\simeq} U(2)$$

sending $[(A,z)]$ to zA, we can then also put: $\mathrm{Spin}^c(3) = U(2)$.

REMARK 3.8. The canonical homomorphism $\beta : \mathrm{Spin}(3) \to \mathrm{Spin}^c(3)$ (§3.1.5) corresponds to the usual inclusion $SU(2) \hookrightarrow U(2)$, while the short exact sequence involving the groups $U(1)$, $\mathrm{Spin}^c(3)$ and $SO(3)$ can be rewritten as:

$$1 \longrightarrow U(1) \xrightarrow{j} U(2) \xrightarrow{\rho} SO(3) \longrightarrow 1,$$

where $j(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ and where $\rho(A) = \pi\left(\frac{1}{\sqrt{\det(A)}}\cdot A\right)$. Finally, the canonical homomormorphism $\mathrm{Spin}^c(3) \to U(1)$ (§3.1.6) is the determinant map $\det : U(2) \to U(1)$.

[4]Recall that the left multiplicative notation is used for affine actions.

LEMMA 3.9. *The following diagram is commutative:*

$$\begin{tikzcd} SO(2) \arrow[r, "\simeq"] \arrow[d, hook] & U(1) \arrow[r, hook] & U(2) \arrow[d, "\rho"] \\ SO(3) \arrow[rr, equal] & & SO(3), \end{tikzcd}$$

Here, $SO(2)$ embeds into $SO(3)$ by $A \mapsto (1) \oplus A$, and $U(1)$ into $U(2)$ by: $A \mapsto A \oplus (1)$.

PROOF. It is a short, straightforward calculation. □

3.2.2. *Relative* Spin^c*-structures in dimension* 3. In the previous section, rigid structures have naturally appeared as representatives for some homotopy classes. However, rigid Spin^c-structures and so, relative Spin^c-structures as they were defined, are not a natural object from a geometric-topological viewpoint. In dimension 3, this can be partially corrected.

THEOREM 3.10. *Let M be a 3-manifold with boundary and let $s_0, s_1 \in \mathrm{Spin}_r(\partial M)$ represent the same spin structure: $[s_0] = [s_1] \in \mathrm{Spin}(\partial M)$. Then, the rigid Spin^c-structures $\beta(s_0)$ and $\beta(s_1)$ can be extended to M and there exists a canonical bijection:*

$$\mathrm{Spin}^c(M, \beta s_0) \xrightarrow{\rho_{s_0,s_1}} \mathrm{Spin}^c(M, \beta s_1),$$

which is $H^2(M, \partial M)$-equivariant.

DEFINITION 3.11. Theorem 3.10 allows us to associate to any 3-manifold M with boundary and any $\sigma \in \mathrm{Spin}(\partial M)$, the space of Spin^c*-structures on M relative to σ*. Denoted by:

$$\mathrm{Spin}^c(M, \sigma),$$

it is an affine $H^2(M, \partial M)$-space.

EXAMPLE 3.12. Let us consider the particular case when ∂M is a disjoint union of tori. The torus $\mathbf{T}^2$ has a *distinguished* Spin-structure: that one which is induced by its Lie group structure, we denote it by σ_0. Thus, ∂M can be given the distinguished Spin-structure $\dot\cup \sigma_0$. It can be verified that the space $\mathrm{Spin}^c(M, \dot\cup\sigma_0)$ is in canonical bijection with the space of relative Spin^c-structures defined by Turaev in [**T6**, §1.2].

PROOF OF THEOREM 3.10. Let $w_2(M, s_i) \in H^2(M, \partial M; \mathbb{Z}_2)$ denote the obstruction to extending s_i to a rigid spin structure on M. It satisfies:

$$\beta(w_2(M, s_i)) = w(M, \beta(s_i)) \in H^3(M, \partial M),$$

where $w(M, \beta(s_i))$ is the relative obstruction defined by Proposition 3.5. Thus, $w(M, \beta(s_i))$ is of order at most 2 and so vanishes since $H^3(M, \partial M)$ is torsion free.

We now prove the second statement. Let $(s_t)_{t \in I}$ denote a homotopy between s_0 and s_1: s_t is a bundle morphism $\tau_{\partial M} \to \gamma_{\mathrm{Spin}}$. Such a homotopy defines a rigid spin structure $s : \tau_{\partial M \times I} \to \gamma_{\mathrm{Spin}}$ over $\partial M \times I$ by putting:

$$s|_{\tau_{\partial M \times I}|_{\partial M \times t}} := s_t,$$

where $\tau_{\partial M \times I}|_{\partial M \times t}$ is identified with $\tau_{\partial M}$. This defines a map:

$$\mathrm{Spin}^c(M, \beta s_0) \xrightarrow{\rho_s} \mathrm{Spin}^c(M, \beta s_1)$$

by the gluing formula: $\rho_s([u]) = [u \cup \beta s]$, where u is a rigid Spin^c-structure on M extending $\beta(s_0)$ (here we are identifying M with its "collar growing" $M \cup (\partial M \times I)$). Observe that ρ_s is $H^2(M, \partial M)$-equivariant.

It suffices now to prove that if $(s'_t)_t$ is another homotopy between s_0 and s_1, then $\rho_s = \rho_{s'}$. Putting $\rho_{s_0,s_1} := \rho_s$ will give the theorem. The image of:

$$\mathrm{Spin}\left(\partial M \times I, (-s_0) \times 0 \cup s_1 \times 1\right) \xrightarrow{\beta} \mathrm{Spin}^c\left(\partial M \times I, (-\beta s_0) \times 0 \cup (\beta s_1) \times 1\right)$$

is a singleton since it is affine over $\beta : H^1(\partial M \times I, \partial M \times \partial I; \mathbb{Z}_2) \to H^2(\partial M \times I, \partial M \times \partial I)$ which is trivial (its codomain is isomorphic to the free Abelian group $H_1(\partial M)$). It then follows that the rigid Spin^c-structures βs and $\beta s'$ on $\partial M \times I$ are homotopic rel $\partial M \times \partial I$, so $\rho_s = \rho_{s'}$. □

REMARK 3.13. Let M be a 3-manifold with boundary and let $\sigma \in \mathrm{Spin}(\partial M)$. We define:

$$\mathrm{Spin}(M, \sigma) := \left\{\alpha \in \mathrm{Spin}(M) \ : \ \alpha|_{\partial M} = \sigma\right\}.$$

One can show that there exists a canonical map:

$$\mathrm{Spin}(M, \sigma) \xrightarrow{\beta} \mathrm{Spin}^c(M, \sigma),$$

which is induced by the map $\mathrm{Spin}(M, s) \to \mathrm{Spin}^c\left(M, \beta(s)\right)$, defined in §3.1.5, for an arbitrary $s \in \mathrm{Spin}_r(\partial M)$ representing σ.

3.2.3. Spin^c*-structures as vector fields: the closed case.* We now review the geometric Euler structures introduced by Turaev in [**T4**]. In this paragraph, we fix a closed 3-manifold M.

DEFINITION 3.14. A *geometric Euler structure on M* is a nonsingular vector field tangent to M, up to punctured homotopy. Precisely, two nonsingular vector fields v and v' on M are considered as equivalent, when there exists a point $x \in M$ such that the restrictions of v and v' to $M \setminus x$ are homotopic among nonsingular vector fields on $M \setminus x$.

If a cellular decomposition of M is given, punctured homotopy coincides with homotopy on the 2-skeleton of M. From obstruction theory, we then deduce that geometric Euler structures exist (since $\chi(M) = 0$) and that they form an affine $H^2(M)$-space denoted by:

$$\mathrm{Vect}(M).$$

The affine action is denoted multiplicatively.

LEMMA 3.15 (Turaev, [**T5**]). *There exists a canonical $H^2(M)$-equivariant bijection:*

$$\mathrm{Vect}(M) \xrightarrow{h_M} \mathrm{Spin}^c(M).$$

PROOF. Let v be a nonsingular vector field tangent to M. Endow M with a Riemannian metric. First, v determines a reduction of F_M to $SO(2)$ with respect to this injection $SO(2) \hookrightarrow SO(3)$ featured by Lemma 3.9: this reduction is $F_{v^\perp}$, the bundle of oriented orthonormal frames of the oriented vector bundle $v^\perp$, when this one is oriented with the right-hand rule (v=right thumb). Then, by the the homomorphism $SO(2) \to U(2)$ of Lemma 3.9, $F_{v^\perp}$ defines a principal $U(2)$-bundle η. According to that lemma, this $U(2)$-bundle can be accompanied with an isomorphism of principal $SO(3)$-bundles $H : \eta/U(1) \to F_M$, and so defines (according to Lemma 3.3) a Spin^c-structure on M. That one only depends on the punctured homotopy class of v: so is defined $h_M([v])$. Moreover, one can verify that the assignation $([v] \mapsto h_M([v]))$ is $H^2(M)$-equivariant; in particular, it is bijective. □

There is a noteworthy involution of $\mathrm{Vect}(M)$. Called *inversion* and denoted by $\xi \mapsto \xi^{-1}$, it is defined by $\xi^{-1} = [-v]$ if $\xi = [v]$.

LEMMA 3.16. *For any $\xi \in \mathrm{Vect}(M)$, we have:*

$$c\left(h_M(\xi)\right) = \frac{\xi}{\xi^{-1}} \in H^2(M).$$

PROOF. Both of the maps $\mathrm{Vect}(M) \to H^2(M)$ which are defined by $\xi \mapsto \xi/\xi^{-1}$ and by $\xi \mapsto c(h_M(\xi))$, are affine over the square[5] map (according to [**T4**, Th. 5.3.1] and Remark 3.7). It then suffices to show the following implication:

$$\left(c\left(h_M(\xi)\right) = 1 \right) \implies \left(\xi/\xi^{-1} = 1 \right). \tag{3.1}$$

Give M a Riemannian metric. According to the last statement of Remark 3.8, the isomorphism class of principal $U(1)$-bundles defined by the Spin^c-structure $h_M(\xi)$, is represented by $F_{v^\perp}$, so that $c\left(h_M(\xi)\right) = c_1\left(F_{v^\perp}\right)$. Consequently, the class $c\left(h_M(\xi)\right)$ is the obstruction to find a nonsingular section of T_M transverse to v. Assertion (3.1) then follows. □

In the latter proof and by Remark 3.7, it appears that the image of $\mathrm{Spin}(M)$ in $\mathrm{Spin}^c(M)$ correspond to the Euler structures $\xi = [v]$ for which there exists a nonsingular tangent vector field to M transverse to v. Let us make this fact more precise. We denote by:

$$\mathrm{Parall}(M)$$

the set of parallelizations of M, up to punctured homotopy. A parallelization of M is a trivialization $e = (e_1, e_2, e_3)$ of the oriented vector bundle T_M.
Let V be a manifold of dimension $n \geq 3$. The space of trivializations of T_V on the 1-skeleton of V, which can be extended to the 2-skeleton, and considered up to homotopy, is empty or is parametrized by $H^1(M;\mathbb{Z}_2)$. It can also be put in a canonical correspondance with $\mathrm{Spin}(M)$ (see [**Ki**, Chap. 4]). Since $\pi_2(SO(3)) = 0$ and since M is of dimension 3, we obtain the following lemma which can be proved similarly to Lemma 3.15.

LEMMA 3.17. *There exists a canonical and $H^1(M;\mathbb{Z}_2)$-equivariant bijection:*

$$\mathrm{Parall}(M) \xrightarrow{h_M} \mathrm{Spin}(M).$$

We define a map:

$$\mathrm{Parall}(M) \xrightarrow{\beta} \mathrm{Vect}(M).$$

by putting $\beta([e]) = [e_1]$, if $e = (e_1, e_2, e_3)$ is a parallelization of M. It then follows from the various definitions that:

LEMMA 3.18. *The following diagram is commutative:*

$$\begin{tikzcd} \mathrm{Parall}(M) \arrow[r, "h_M"] \arrow[d, "\beta"'] & \mathrm{Spin}(M) \arrow[d, "\beta"] \\ \mathrm{Vect}(M) \arrow[r, "h_M"'] & \mathrm{Spin}^c(M). \end{tikzcd}$$

where the map $\beta : \mathrm{Spin}(M) \to \mathrm{Spin}^c(M)$ has been defined in §3.1.5.

[5] Recall that the left multiplicative notation is used for affine actions.

3.2.4. Spin^c*-structures as vector fields: the boundary case.* We now want to define some relative geometric Euler structures.

On the hand, all of the definitions and results of §3.2.3 are relative to T_M and work for any 3-dimensional oriented vector bundle. In particular, if S is a closed surface, they can be applied to $\epsilon^1 \oplus T_S$, where ϵ^i stands for the i-dimensional trivial vector bundle. Thus, we define:

$$\operatorname{Vect}(S_{st}) \quad \text{and} \quad \operatorname{Parall}(S_{st}),$$

to be respectively the set of nonsingular sections of $\epsilon^1 \oplus T_S$ and the set of trivializations of this oriented vector bundle, all of them being considered up to homotopy in S. Then, obstruction theory says that they are respectively affine spaces over $H^2(S)$ and $H^1(S;\mathbb{Z}_2)$. Some analogs of Lemma 3.15 and Lemma 3.17 state the existence of canonical affine isomorphisms:

$$\operatorname{Vect}(S_{st}) \xrightarrow{h_S} \operatorname{Spin}^c(S) \qquad \text{and} \qquad \operatorname{Parall}(S_{st}) \xrightarrow{h_S} \operatorname{Spin}(S).$$

Finally, an analog of Lemma 3.18 states that they fit both together in a commutative diagram.

Example 3.19. The canonical section $v = (1,0)$ of $\epsilon^1 \oplus T_S$ determines a Spin^c-structure $h_S([v])$ on S. Its Chern class is equal to the Euler class $e(S)$ of the surface S (since here $v^\perp = T_S \leq \epsilon^1 \oplus T_S$ for a product metric).

On the other hand, one can speak of rigid structures for any kind of structures which are defined as homotopy classes of something. Thus, there are rigid versions of $\operatorname{Vect}(N)$ and $\operatorname{Parall}(N)$, when $N = M$ is a 3-manifold or when $N = S_{st}$ with S a closed surface. These rigid versions are denoted with a decorating subscript "r". Thus, if M is a 3-manifold with non-empty boundary and if $v \in \operatorname{Vect}_r((\partial M)_{st})$ is a nonsingular section of $\epsilon^1 \oplus T_{\partial M} = T_M|_{\partial M}$, one can define:

$$\operatorname{Vect}(M, v)$$

the space of *geometric Euler structures on M relative to v*. This space is empty or is an affine $H^2(M, \partial M)$-space. In exactly the same way we obtained Theorem 3.10, we get:

Theorem 3.20. *Let M be a 3-manifold with non-empty boundary and let $e = (e_1, e_2, e_3)$, $e' = (e'_1, e'_2, e'_3)$ be some homotopic trivializations of $T_M|_{\partial M}$. Then, the nonsingular sections e_1 and e'_1 of $T_M|_{\partial M}$ can be extended to M, and there exists a canonical bijection:*

$$\operatorname{Vect}(M, e_1) \xrightarrow{\rho_{e,e'}} \operatorname{Vect}(M, e'_1),$$

which is $H^2(M, \partial M)$-equivariant.

Definition 3.21. Theorem 3.10 allows us to associate to any 3-manifold with boundary and any $\rho \in \operatorname{Parall}\big((\partial M)_{st}\big)$, the space of *geometric Euler structures on M relative to ρ*. Denoted by:

$$\operatorname{Vect}(M, \rho),$$

it is an affine $H^2(M, \partial M)$-space.

Hence this relative version of Lemma 3.15 which may be proved similarly:

Lemma 3.22. *Let M be a 3-manifold with boundary and let $\rho \in \operatorname{Parall}((\partial M)_{st})$. There exists then a canonical and $H^2(M, \partial M)$-equivariant bijection:*

$$\operatorname{Vect}(M, \rho) \longrightarrow \operatorname{Spin}^c(M, h_{\partial M}(\rho)).$$

3.2.5. *Relative Chern classes.* We now define some relative versions of the Chern classes of Spin^c-structures (§3.1.6).

LEMMA 3.23. *Let M be a 3-manifold with boundary and let $\sigma \in \mathrm{Spin}(\partial M)$. There exists then a canonical map:*

$$\mathrm{Spin}^c(M,\sigma) \xrightarrow{c} H^2(M,\partial M),$$

which is affine over the square[6] *map defined by $x \mapsto x^2$.*

DEFINITION 3.24. If $\alpha \in \mathrm{Spin}^c(M,\sigma)$, $c(\alpha)$ is called the *Chern class of the relative* Spin^c*-structure* α.

PROOF OF LEMMA 3.23. Let $\rho \in \mathrm{Parall}\left((\partial M)_{st}\right)$ correspond to σ by $h_{\partial M}$. In fact, we will define a map $c : \mathrm{Vect}(M,\rho) \to H^2(M,\partial M)$ (and then apply Lemma 3.22).

Let $e = (e_1, e_2, e_3)$ be a trivialization of $\epsilon^1 \oplus T_{\partial M}$ representing ρ and let $\xi \in \mathrm{Vect}(M, e_1)$ be represented by v: v is a nonsingular tangent vector field to M whose restriction to ∂M coincides with e_1. The vector field e_2 is a nonsingular section of T_M on ∂M transverse to v. We then define $c(\xi)$ to be the first obstruction to extending e_2 to a nonsingular section of T_M transverse to v; we are then led to a map $\mathrm{Vect}(M,e_1) \to H^2(M,\partial M)$. If e' is another representant of ρ, the analogously obtained map $\mathrm{Vect}(M,e_1') \to H^2(M,\partial M)$ coincide with the latter one *via* the isomorphism $\rho_{e,e'}$ involved in Th. 3.20. We then get a well-defined map c.

Let now $x \in H^2(M,\partial M)$ and suppose that its Poincaré dual $P^{-1}x \in H_1(M)$ is represented by a smooth oriented knot $L \subset \mathrm{int}(M)$. Then, just as in the closed case, it can be shown that the geometric Euler structure $x \cdot \xi$ is represented by the vector field w obtained from v by *Reeb turbulentization* along L (see [**T4**, §5.2] or [**T8**, §1.1]). It is then not difficult to prove, by a direct calculus of obstruction in the oriented 3-manifold M using this concrete description of w, that $c([w]) = x^2 \cdot c([v])$. □

REMARK 3.25. For any $\alpha \in \mathrm{Spin}^c(M,\sigma)$, the Chern class $c(\alpha)$ vanishes if and only if α comes from $\mathrm{Spin}(M,\sigma)$ (Remark 3.13).

We now give a modulo 2 formula for these relative Chern classes. Recall that the cobordism group Ω_1^{Spin} is isomorphic to $\mathbb{Z}_2$: the generator is $\mathbf{S}^1$ endowed with the Spin-structure which is induced by its Lie group structure (see [**Ki**, p. 35, 36]). For a closed oriented surface S, Johnson constructs in [**J1**] a canonical bijection:

$$\mathrm{Spin}(S) \underset{\simeq}{\xrightarrow{q}} \mathrm{Quad}(\bullet)$$

between spin structures on M and quadratic forms over the modulo 2 intersection pairing $\bullet$ of S. For any $\sigma \in \mathrm{Spin}(S)$, the quadratic form $q_\sigma : H_1(S;\mathbb{Z}_2) \to \mathbb{Z}_2$ is defined as follows. If γ is an oriented simple closed curve on S, we have:

$$q_\sigma\left([\gamma]\right) = \left[(\gamma, \sigma|_\gamma)\right] \in \Omega_1^{\mathrm{Spin}} \simeq \mathbb{Z}_2.$$

LEMMA 3.26. *Let M be a 3-manifold with boundary, $\sigma \in \mathrm{Spin}(\partial M)$ and $\alpha \in \mathrm{Spin}^c(M,\sigma)$. Then:*

$$\forall y \in H_2(M,\partial M), \quad \langle c(\alpha), y\rangle = q_\sigma\left(\partial_*(y)\right) \ mod\ 2,$$

where $\langle\cdot,\cdot\rangle$ denotes Kronecker evaluation, and where $\partial_* : H_2(M,\partial M) \to H_1(\partial M)$ is the connecting homomorphism of the pair $(M,\partial M)$.

[6] Recall that the left multiplicative notation is used for affine actions.

PROOF. The modulo 2 reduction of $c(\alpha)$ is:

$$w_2(M,\sigma) \in H^2(M,\partial M;\mathbb{Z}_2),$$

which is the obstruction to extending σ to the whole manifold M. Let Σ be a connected immersed surface in M such that $\partial\Sigma = \partial M \cap \Sigma$, $\partial\Sigma$ has no singularity and Σ represents the modulo 2 reduction of y. Then, $\langle c(\alpha), y\rangle = \langle w_2(M,\sigma), [\Sigma]\rangle$ mod 2 is equal to $\langle w_2\left(\Sigma, \sigma|_{\partial\Sigma}\right), [\Sigma]\rangle$ and so is the obstruction to extending the Spin-structure $\sigma|_{\partial\Sigma}$ to the whole surface Σ. Since Σ is connected, this is the class of $(\partial\Sigma, \sigma|_{\partial\Sigma})$ in Ω_1^{Spin}. Thus, $\langle c(\alpha), y\rangle = q_\sigma\left([\partial\Sigma]\right) = q_\sigma\left(\partial_*(y)\right)$ modulo 2. $\square$

EXAMPLE 3.27. Suppose that M is a 3-manifold with a disjoint union of tori as boundary. We denote by $\rho_0 \in \mathrm{Parall}\left((\mathbf{T}^2)_{st}\right)$ this distinguished parallelization of the torus corresponding to its distinguished Spin-structure σ_0 (Example 3.12). Then, the geometric Euler structures on M relative to $\dot\cup\rho_0$, as introduced in Definition 3.21, correspond to the relative geometric Euler structures of Turaev (see [**T4**, §5.1] or [**T8**, §1.1]). In particular, Lemma 3.26 is a generalization of [**T8**, Lemma 1.3].

3.3. Gluing of Spin^c**-structures.** We are now ready to deal with the technical problem of gluing Spin^c-structures.

Let us consider two oriented compact smooth n-manifolds M_1 and M_2, together with a positive diffeomorphism $f : -\partial M_2 \to \partial M_1$. Form the closed manifold:

$$M = M_1 \ \cup_f \ M_2.$$

The inclusion $M_i \hookrightarrow M$ will be denoted by j_i.

LEMMA 3.28. *Let some rigid structures $s_1 \in \mathrm{Spin}^c_r(\partial M_1)$ and $s_2 \in \mathrm{Spin}^c_r(\partial M_2)$ be such that $f^*(s_1) = -s_2$. We suppose that the relative obstruction $w(M_i, s_i) \in H^3(M,\partial M)$ vanishes for $i = 1$ and 2. Then, the absolute obstruction $w(M)$ vanishes and there exists a gluing map:*

$$\mathrm{Spin}^c(M_1,s_1) \times \mathrm{Spin}^c(M_2,s_2) \xrightarrow{\cup_f} \mathrm{Spin}^c(M),$$

which is affine over:

$$\begin{tikzcd}
H^2(M_1,\partial M_1) \oplus H^2(M_2,\partial M_2) \arrow[r, dotted] \arrow[d, "P^{-1}\times P^{-1}"'] & H^2(M) \\
H_{n-2}(M_1) \oplus H_{n-2}(M_2) \arrow[r, "j_{1,*}\oplus j_{2,*}"] & H_{n-2}(M) \arrow[u, "P"']
\end{tikzcd}$$

where the letter P stands for a Poincaré duality isomorphism.

PROOF. Let $\alpha_i \in \mathrm{Spin}^c(M_i, s_i)$ be represented by a rigid structure a_i. Then, a_1 and a_2 can be glued by means of f: we obtain a rigid Spin^c-structure on M whose homotopy class does not depend on the choices of a_1 and a_2 in their respective classes α_1 and α_2. We denote it by:

$$\alpha_1 \cup_f \alpha_2 \in \mathrm{Spin}^c(M).$$

Let us prove that the so obtained map $\cup_f$ is affine.

The manifolds M_1 and M_2 being smooth, they are triangulable. Let $\mathcal{C}_i$ be a triangulation of M_i for $i = 1$ and 2, such that $\mathcal{C}_1|_{\partial M_1}$ corresponds to $\mathcal{C}_2|_{\partial M_2}$ by f. We denote by $\mathcal{C}_i^*$ the cellular decomposition of M_i dual to the triangulation $\mathcal{C}_i$.
On the one hand, we consider the *union* $\mathcal{C}$ of the triangulations $\mathcal{C}_1$ and $\mathcal{C}_2$: a simplex of $\mathcal{C}$ is a simplex of $\mathcal{C}_i$ for $i = 1$ or 2, and simplices on ∂M_1 are identified with simplices on ∂M_2 by f. On the other hand, we consider the *gluing* $\mathcal{C}^*$ of the cellular decompositions $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$: a cell of $\mathcal{C}^*$ either is a cell of $\mathcal{C}_i^*$ which does not intersect ∂M_i, either is the gluing by f of a cell belonging to $\mathcal{C}_1^*$ with a cell of

$\mathcal{C}_2^*$ along a face lying in ∂M_i. Then, $\mathcal{C}$ is a triangulation of M and $\mathcal{C}^*$ is its dual cellular decomposition. Cohomology will be calculated with $\mathcal{C}$ while homology will be computed with $\mathcal{C}^*$.
Let $\alpha_i, \alpha_i' \in \mathrm{Spin}^c(M_i, s_i)$ $(i=1,2)$, $\alpha = \alpha_1 \cup_f \alpha_2$ and $\alpha' = \alpha_1' \cup_f \alpha_2'$. We want to prove this equality:

$$j_{1,*}P^{-1}\left(\frac{\alpha_1}{\alpha_1'}\right) + j_{2,*}P^{-1}\left(\frac{\alpha_2}{\alpha_2'}\right) = P^{-1}\left(\frac{\alpha}{\alpha'}\right) \in H_{n-2}(M). \tag{3.2}$$

Let $a_i, a_i' \in \mathrm{Spin}^c_r(M_i)$ be some respective representatives for α_i and α_i' which coincides on the 1-skeleton of $\mathcal{C}_i$ (and, of course, on ∂M_i). We have fixed a bundle morphism $\tau_{M_i} \to \gamma_{SO}$ and, as in the proof of Prop. 3.2, we identify the rigid structure a_i and a_i' with some lifts $M_i \to B\mathrm{Spin}^c$ of the corresponding map $M_i \to BSO$ at the level of base spaces. Then $\alpha_i/\alpha_i' \in H^2(M_i, \partial M_i)$ is the class of the 2-cocycle which assigns to each 2-cell e^i_k of $\mathcal{C}_i$ outside ∂M_i, this element z^i_k of $\pi_2(BU(1)) \simeq \pi_2(K(\mathbb{Z},2)) \simeq \mathbb{Z}$ obtenaid by gluing $a_i|_{e^i_k}$ and $a_i'|_{e^i_k}$ along ∂e^i_k. Then, $P^{-1}(\alpha_i/\alpha_i') = \left[\sum_k z^i_k \cdot e^{*,i}_k\right]$ where $e^{*,i}_k$ denotes the $(n-2)$-cell dual to e^i_k.
Moreover, $a := a_1 \cup_f a_2$ and $a' := a_1' \cup_f a_2'$ respectively represent α and α'. Similarly, using these rigid structures, we can describe explicitly a 2-cocycle representing α/α'. This 2-cocycle sends any 2-cell of $\mathcal{C}_1 \cup_f \mathcal{C}_2$ contained in $\partial M_1 \cong -\partial M_2$ to $0 \in \mathbb{Z}$ so that $P^{-1}(\alpha/\alpha')$ is represented by $\sum_k z^1_k \cdot e^{*,1}_k + \sum_k z^2_k \cdot e^{*,2}_k$. □

We now come back to the dimension $n = 3$. Here is the gluing lemma which we will practice.

COROLLARY 3.29. *Let $\sigma_1 \in \mathrm{Spin}(\partial M_1)$ and $\sigma_2 \in \mathrm{Spin}(\partial M_2)$ be such that $f^*(\sigma_1) = -\sigma_2$. There exists then a canonical gluing map:*

$$\mathrm{Spin}^c(M_1, \sigma_1) \times \mathrm{Spin}^c(M_2, \sigma_2) \xrightarrow{\cup_f} \mathrm{Spin}^c(M),$$

which is affine over:

$$\begin{tikzcd}
H^2(M_1, \partial M_1) \oplus H^2(M_2, \partial M_2) \arrow[r, dotted] \arrow[d, "P^{-1} \times P^{-1}"'] & H^2(M) \\
H_1(M_1) \oplus H_1(M_2) \arrow[r, "j_{1,*} \oplus j_{2,*}"] & H_1(M) \arrow[u, "P"']
\end{tikzcd}$$

Furthermore, if $\alpha_i \in \mathrm{Spin}^c(M, \sigma_i)$ $(i=1,2)$, the following identity between Chern classes holds:

$$P^{-1}c(\alpha_1 \cup_f \alpha_2) = j_{1,*}P^{-1}c(\alpha_1) + j_{2,*}P^{-1}c(\alpha_2) \in H_1(M).$$

PROOF. According to Th. 3.10, the relative obstructions vanish: the first statement is then a direct application of Lemma 3.28 and definition of Spin^c-structures relative to Spin-structures. The second statement is again a calculus of gluing of obstructions in oriented manifolds and its proof uses the same arguments as in the proof of Lemma 3.28. □

REMARK 3.30. If M is obtained from M_1 and M_2 by gluing them along only part of their boundaries (so that $\partial M \neq \varnothing$), there are relative versions of Lemma 3.28 and Corollary 3.29 (involving Spin^c-structures on M relative to Spin-structures).

3.4. Combinatorial descriptions associated to surgery presentations in $\mathbf{S}^3$. In this paragraph, we fix an ordered oriented framed n-component link L in $\mathbf{S}^3$ and we denote by $B_L = (b_{ij})_{i,j=1,\ldots,n}$ the linking matrix of L. We call V_L the

3-manifold obtained from $\mathbf{S}^3$ by surgery along L and we denote by W_L the *trace* of the surgery. So:

$$V_L = \partial W_L \quad \text{with:} \quad W_L = \mathbf{D}^4 \cup \bigcup_{i=1}^{n} \left(\mathbf{D}^2 \times \mathbf{D}^2\right)_i ,$$

where the 2-handle $\left(\mathbf{D}^2 \times \mathbf{D}^2\right)_i$ is attached by embedding $-\left(\mathbf{S}^1 \times \mathbf{D}^2\right)_i$ into $\mathbf{S}^3 = \partial \mathbf{D}^4$ in accordance with the specified framing and orientation of L_i.
The group $H_2(W_L)$ is then free Abelian of rank n. It is given the *preferred* basis $([S_1], \ldots, [S_n])$. Here, the closed surface S_i is taken to be $\left(\mathbf{D}^2 \times 0\right)_i \cup (-\Sigma_i)$, where Σ_i is a Seifert surface for L_i in $\mathbf{S}^3$ which has been push-off into the interior of $\mathbf{D}^4$ as shown in Figure 3.1. Also, $H^2(W_L)$ will be identified with $\mathrm{Hom}(H_2(W_L), \mathbb{Z})$ by

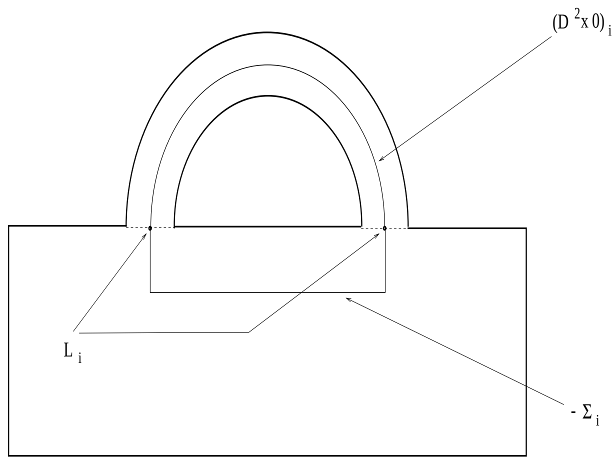

Figure 3.1: The preferred basis of $H_2\,(W_L)$.

Kronecker evaluation, and will be given the dual basis. In the sequel, we simplify our notation by putting $H = H_2(W_L)$ (so that $H^2(W_L)$ is identified with H^*) and by denoting $f : H \times H \to \mathbb{Z}$ the intersection pairing of W_L; recall that the matrix of f relatively to the preferred basis of H is B_L. Since (H, f) is a symmetric bilinear lattice, it can be applied the results of §2.2.

3.4.1. *Combinatorial description of Spin-structures.* We begin by recalling a combinatorial description of $\mathrm{Spin}(V_L)$ due to Blanchet ([**Bl**]). Define the set:

$$\mathcal{S}_L = \left\{ [r] = ([r_i])_{i=1}^n \in (\mathbb{Z}_2)^n \ : \ \forall i = 1, \ldots, n, \ \sum_{j=1}^{n} b_{ij} r_j \equiv b_{ii} \bmod 2 \right\}.$$

The elements of $\mathcal{S}_L$ are called *characteristic solutions of* B_L. Here, $\mathcal{S}_L$ is refered to as the *combinatorial description of* $\mathrm{Spin}(V_L)$, as justified by the following lemma.

LEMMA 3.31. *There are canonical bijections:*

$$\mathrm{Spin}(V_L) \xrightarrow[\simeq]{} \mathrm{Wu}(f)/2H \xrightarrow[\simeq]{} \mathcal{S}_L .$$

A refined Kirby's theorem dealing with surgery presentations of closed Spin-manifolds of dimension 3 can then be derived from this lemma (see [**Bl**, Th. (I.1)]).

PROOF OF LEMMA 3.31. The preferred basis of H induces an isomorphism $H/2H \simeq (\mathbb{Z}_2)^n$: the bijection between $\mathrm{Wu}(f)/2H$ and $\mathcal{S}_L$ is obtained by this way.

We now describe a bijection between $\mathrm{Spin}(V_L)$ and $\mathrm{Wu}(f)/2H$. Let σ be a Spin-structures on V_L, the obstruction $w_2(W_L, \sigma)$ σ to W_L belongs to the group $H^2(W_L, V_L; \mathbb{Z}_2) \simeq H_2(W_L; \mathbb{Z}_2) \simeq H/2H$. Since $w_2(W_L, \sigma)$ goes to the second Stiefel-Whitney class $w_2(W_L)$ by the restriction map $H^2(W_L, V_L; \mathbb{Z}_2) \to H^2(W_L; \mathbb{Z}_2)$, a representative for $w_2(W_L, \sigma)$ in H has to be a Wu class for f. □

3.4.2. *Combinatorial description of* Spin^c*-structures.* We now give a Spin^c analogue of the previous paragraph. For this, we define the set:

$$\mathcal{S}^c_L = \frac{\{s = (s_i)_{i=1}^n \in \mathbb{Z}^n \ : \ \forall i = 1, \ldots, n, \ s_i \equiv b_{ii} \bmod 2\}}{2 \cdot \mathrm{Im}\ B_L},$$

which is referred to here as the *combinatorial description of* $\mathrm{Spin}^c(V_L)$. Indeed, we have:

LEMMA 3.32. *There are canonical bijections:*

$$\mathrm{Spin}^c(V_L) \xrightarrow[\simeq]{} \mathrm{Char}(f)/2 \cdot \mathrm{Im}\ \widehat{f} \xrightarrow[\simeq]{} \mathcal{S}^c_L.$$

PROOF. The preferred basis of H defines an isomorphism $H^* \simeq \mathbb{Z}^n$, which induces a bijection between $\mathrm{Char}(f)/2 \cdot \mathrm{Im}\widehat{f}$ and $\mathcal{S}^c_L$.
The restriction map $\mathrm{Spin}^c(W_L) \to \mathrm{Spin}^c(V_L)$ is affine over the map $H^2(W_L) \to H^2(V_L)$ induced by inclusion. By exactness of the pair (W_L, V_L), the latter is surjective and its kernel coincides with the image of $\widehat{f} : H \to H^*$ (by Poincaré duality). Moreover, since $H^2(W_L)$ is free Abelian, a Spin^c-structure α on W_L is determined by its Chern class $c(\alpha) \in H^2(W_L) \simeq H^*$. Such a form has to be a characteristic form for f since its modulo 2 reduction coincides with the second Stiefel-Whitney class $w_2(W_L) \in H^2(W_L; \mathbb{Z}_2) \simeq \mathrm{Hom}(H, \mathbb{Z}_2)$. There is then a bijection between $\mathrm{Spin}^c(V_L)$ and $\mathrm{Char}(f)/2 \cdot \mathrm{Im}\widehat{f}$ defined by $\sigma \mapsto c(\tilde{\sigma})$ where $\tilde{\sigma}$ is an extension of σ to W_L. □

Concretely, in a surgery diagram for the Spin^c-manifold (V_L, σ), we draw the framed link L with the blackboard framing convention and we add a decorating integer s_i to each component L_i of L so that the multi-integer $s = (s_i)$ is a combinatorial representant for the Spin^c-structure σ.
We now give the corresponding refined Kirby's theorem dealing with surgery presentations of closed Spin^c-manifolds of dimension 3.

THEOREM 3.33. *Let L and L' be some ordered oriented framed links in $\mathbf{S}^3$, and let $[s] \in \mathcal{S}^c_L$ and $[s'] \in \mathcal{S}^c_{L'}$ correspond respectively to $\sigma \in \mathrm{Spin}^c(V_L)$ and $\sigma' \in \mathrm{Spin}^c(V_{L'})$. Then, the manifolds (V_L, σ) and $(V_{L'}, \sigma')$ are* Spin^c*-diffeomorphic if, and only if, the pairs $(L, [s])$ and $(L', [s'])$ are related one to another by a finite sequence of the moves depicted on Figure 3.2.*

PROOF. This follows from the usual Kirby's theorem. It suffices to show that, for each Kirby's move $L_1 \to L_2$, the corresponding canonical diffeomorphism $V_{L_1} \to V_{L_2}$ acts at the level of Spin^c-structures as combinatorially described on Figure 3.2. This is a straightforward verification. □

EXAMPLE 3.34. Look at the *slam dunk* move, depicted on Figure 3.3, between surgery presentations in $\mathbf{S}^3$ of closed Spin^c-manifolds. Here, we are considering the ordered union $L \cup (K_1, K_2)$ of a n-component ordered oriented framed link L with a framed oriented knot K_1 together with its meridian K_2. The move is:

$$(L \cup (K_1, K_2), [(s_1, \ldots, s_n, y, 0)]) \longleftrightarrow (L, [(s_1, \ldots, s_n)]),$$

where y is the framing number of K_1. It can be shown by rewriting the proof of [**FR**, Lemma 5] with Spin^c Kirby's calculi.

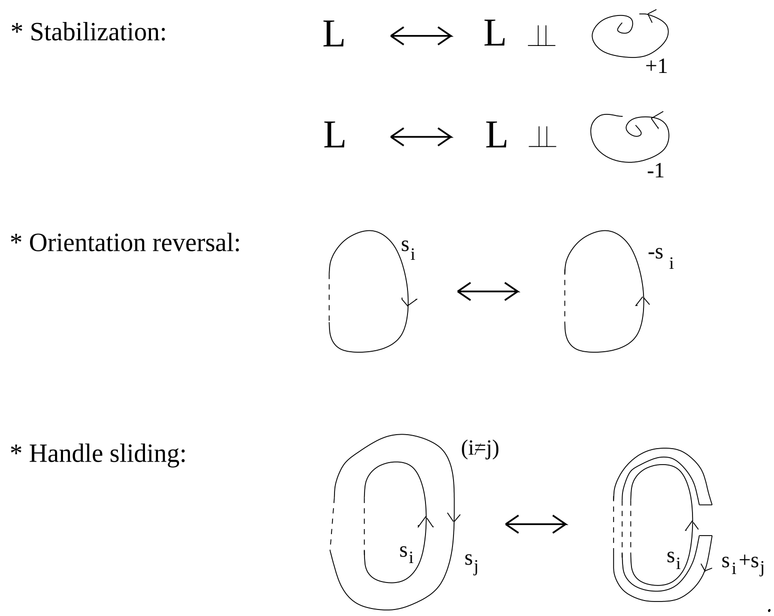

Figure 3.2: Spin^c Kirby's moves

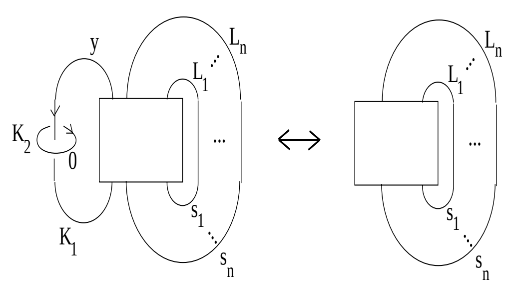

Figure 3.3: A Spin^c-version of the slam dunk move.

Remark 3.35. As follows from the following commutative diagram with exact rows, there is a canonical isomorphism Coker $\widehat{f} = H^*/\mathrm{Im}\ \widehat{f} \to H^2(V_L)$ which we denote by υ:

$$\begin{tikzcd}
H^2(W_L, V_L) \arrow[r] & H^2(W_L) \arrow[r] & H^2(V_L) \arrow[r] & 0 \\
H \arrow[u, "\simeq"', "P"] \arrow[r, "\widehat{f}"] & H^* \arrow[u, "\simeq"'] \arrow[r] & \text{Coker } \widehat{f} \arrow[u, dotted, "\simeq"', "\upsilon"] \arrow[r] & 0
\end{tikzcd}$$

The affine action of $H^2(V_L)$ on Spin$^c(V_L)$ writes then combinatorially:

$$\forall [x] \in \text{Coker } \widehat{f},\ \forall [c] \in \text{Char}(f)/2 \cdot \text{Im } \widehat{f}, \quad [x] \cdot [c] = [c + 2x].$$

Also, the Chern clas map Spin$^c(V_L) \to H^2(V_L)$ is combinatorially described by the map $\text{Char}(f)/2 \cdot \text{Im } \widehat{f} \to \text{Coker } \widehat{f}$ defined by $c([c]) = [c]$.
Therefore, as follows from Lemma 3.16, the involution of Spin$^c(V_L)$ defined in §3.2.3 corresponds to the involution of $\mathcal{S}_L^c \simeq \text{Char}(f)/2 \cdot \text{Im}\widehat{f}$ given by: $[s] \mapsto [-s]$.

3.4.3. *From* Spin *to* Spin^c *in a combinatorial way.* We now relate the combinatorial description of $\mathrm{Spin}(V_L)$ with that of $\mathrm{Spin}^c(V_L)$.

LEMMA 3.36. *The canonical map $\beta : \mathrm{Spin}(V_L) \to \mathrm{Spin}^c(V_L)$ is combinatorially described by the map $\beta : \mathrm{Wu}(f)/2H \to \mathrm{Char}(f)/2 \cdot \mathrm{Im}\ \widehat{f}$ given by $\beta([w]) = \left[\widehat{f}(w)\right]$ or, equivalently, by the map $\beta : \mathcal{S}_L \to \mathcal{S}_L^c$ defined by $\beta\left([r]\right) = [B_L \cdot r]$.*

PROOF. Take $\sigma \in \mathrm{Spin}(V_L)$ and let $r_\sigma \in H^2(W_L, V_L; \mathbb{Z}) \simeq \mathbb{Z}^n$ be an integer representative for the obstruction $w_2\left(W_L, \sigma\right) \in H^2(W_L, V_L; \mathbb{Z}_2) \simeq (\mathbb{Z}_2)^n$ to extending σ to W_L. Let also $\tilde{\sigma} \in \mathrm{Spin}^c(W_L)$ be an extension of $\beta(\sigma) \in \mathrm{Spin}^c(V_L)$. Then, the lemma will follow from the fact that r_σ goes to $c(\tilde{\sigma})$ by the natural map $H^2(W_L, V_L) \to H^2(W_L)$ when $\tilde{\sigma}$ is appropriately choosen with respect to r_σ. It can be proved indirectly as follows. In case when σ can be extended to W_L, this is certainly true: indeed, we can then take $r_\sigma = 0$ and we may choose as $\tilde{\sigma}$ the image by β of the unique extension of σ to W_L, so that $c(\tilde{\sigma}) = 0$. The general case can be reduced to this particular one for the following two reasons. First, it is easily verified that for each Kirby's move $L_1 \to L_2$ between oriented framed links, the induced bijections $\mathcal{S}_{L_1} \to \mathcal{S}_{L_2}$ and $\mathcal{S}^c_{L_1} \to \mathcal{S}^c_{L_2}$, which are respectively described in [**Bl**, Th. (I.1)] and Th. 3.33, are compatible with the maps $\mathcal{S}_{L_k} \to \mathcal{S}^c_{L_k}$ $(k = 1, 2)$ defined by $\beta\left([r]\right) = [B_L \cdot r]$. Second, according to a theorem of Kaplan ([**Ka**]), there exists an oriented framed link L' in $\mathbf{S}^3$ related to L by a finite sequence of Kirby's moves through which $\sigma \in \mathrm{Spin}(V_L)$ goes to $\sigma' \in \mathrm{Spin}(V_{L'})$ such that σ' can be extended to $W_{L'}$. □

4. Quadratic functions associated to 3-manifolds with complex spin structures

Gille has shown in [**Gi**] (see also [**De3**]) that, for a closed oriented 3-manifold, torsion Spin^c-structures are in a canonical one-to-one correspondance with quadratic functions over its torsion linking pairing. We now aim at generalizing this result to all Spin^c-structures, not only those of torsion.

4.1. A 4-dimensional definition.

4.1. A 4-dimensional definition. Let M be a closed oriented connected 3-manifold and let $\sigma \in \mathrm{Spin}^c(M)$. In this paragraph, we define a quadratic function:

$$H_2(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbb{Q}/\mathbb{Z}$$

using a surgery presentation of M. Let us choose an ordered oriented framed link L in $\mathbf{S}^3$ together with a positive diffeomorphism $\psi : V_L \to M$. We maintain the notations fixed in §3.4; recall in particular that (H, f) stands for the bilinear lattice $\Big(H_2(W_L), \text{intersection pairing in } W_L\Big)$ and that it can be applied the results of §2.2.

4.1.1. *A combinatorial description of $H_2(V_L; \mathbb{Q}/\mathbb{Z})$.*

LEMMA 4.1. *There exists a canonical isomorphism:*

$$\frac{H^\sharp}{H} \xrightarrow[\simeq]{\kappa} H_2(V_L; \mathbb{Q}/\mathbb{Z}).$$

Proof. Consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 & 0 \\
0 \arrow[r] & H_2(V_L;\mathbb{Q}/\mathbb{Z}) \arrow[r] & H_2(W_L;\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[u] & H_2(W_L,V_L;\mathbb{Q}/\mathbb{Z}) \arrow[u] \\
0 \arrow[r] & H_2(V_L;\mathbb{Q}) \arrow[r] \arrow[u] & H_2(W_L;\mathbb{Q}) \arrow[r, "c"] \arrow[u, "d"] & H_2(W_L,V_L;\mathbb{Q}) \arrow[u] \\
0 \arrow[r] & H_2(V_L;\mathbb{Z}) \arrow[r] \arrow[u] & H_2(W_L;\mathbb{Z}) \arrow[r] \arrow[u, "a"] & H_2(W_L,V_L;\mathbb{Z}) \arrow[u, "b"] \\
 & & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

$H^\sharp$ is the subspace of $H \otimes \mathbb{Q} = H_2(W_L;\mathbb{Q})$ comprising those $x \in H_2(W_L;\mathbb{Q})$ such that $c(x) \in H_2(W_L,V_L;\mathbb{Q})$ satisfies: $\forall y \in H_2(W_L;\mathbb{Z})$, $c(x) \bullet a(y) \in \mathbb{Z}$, where $\bullet$ is the rational intersection pairing in W_L. So $H^\sharp$ is $c^{-1}b(H_2(W_L,V_L;\mathbb{Z}))$. Seeing $H_2(V_L;\mathbb{Q}/\mathbb{Z})$ as a subgroup of $H_2(W_L;\mathbb{Q}/\mathbb{Z})$, we deduce the announced isomorphism from the map d. □

Recall that the quotient group $H^\sharp/H$ is denoted by G_f in §2.2 and that it can be involved in the short exact sequence (2.4). We now interpret this sequence as a universal coefficients short exact sequence for V_L (in particular, we thus recover the fact that it is split). We denote by B the Bockstein homomorphism associated to the following short exact sequence of coefficients:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Lemma 4.2. *(Using the previous notations) The following diagram*[7] *is commutative:*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \widehat{L_f} \arrow[r] \arrow[d, "\kappa|"', "\simeq"] & G_f \arrow[r] \arrow[d, "\kappa"', "\simeq"] & \mathrm{T}\ \operatorname{Coker} \widehat{f} \arrow[r] \arrow[d, "\upsilon"', "\simeq"] & 0 \\
0 \arrow[r] & H_2(V_L) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] & H_2\Big(V_L;\mathbb{Q}/\mathbb{Z}\Big) \arrow[r, "-P\circ B"'] & T(H^2(V_L)) \arrow[r] & 0,
\end{tikzcd}$$

where the isomorphisms κ and υ are respectively defined in Lemma 4.1 and Remark 3.35.

Proof. Commutativity of the left square is equivalent to the equality:

$$\kappa\left(\operatorname{Ker} \widehat{L_f}\right) = H_2(V_L) \otimes \mathbb{Q}/\mathbb{Z} \subset H_2(V_L;\mathbb{Q}/\mathbb{Z}). \tag{4.1}$$

Recall that $\operatorname{Ker} \widehat{L_f} = \left(\operatorname{Ker} \widehat{f_\mathbb{Q}} + H\right)/H = \left(H_2(V_L;\mathbb{Q}) + H_2(W_L;\mathbb{Z})\right)/H_2(W_L;\mathbb{Z})$ where the groups $H_2(V_L;\mathbb{Q})$ and $H_2(W_L;\mathbb{Z})$ are seen here as subgroups of $H_2(W_L;\mathbb{Q})$ Then, by definition of κ an by the commutative diagram in the proof of Lemma 4.1, (4.1) will follow from the fact that the image of the canonical map $H_2(V_L;\mathbb{Q}) \to$

[7] Be careful of the minus sign.

$H_2(V_L;\mathbb{Q}/\mathbb{Z})$ coincides with that of $H_2(V_L)\otimes\mathbb{Q}/\mathbb{Z}\to H_2(V_L;\mathbb{Q}/\mathbb{Z})$. But this can be deduced from the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & H_2(V_L)\otimes\mathbb{Q} \arrow[r] \arrow[d] & H_2(V_L;\mathbb{Q}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2(V_L)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & H_2(V_L;\mathbb{Q}/\mathbb{Z}) & \\
& 0 & &
\end{tikzcd}$$

We now derive commutativity of the right square from commutativity of the left one in the following way. We start with $x\in \mathrm{T}H_1(V_L)\simeq \mathrm{T}H^2(V_L)$ and we represent it by a 1-cycle X. Let Y be a relative 2-cycle in (W_L,V_L) such that $\partial Y = X$ and let $y=[Y]\in H_2(W_L,V_L)$. Let also $n\in\mathbb{N}^\times$ such that $n\cdot x=0$, and let S be a 2-chain in V_L with boundary $n\cdot X$. We then define a 2-cycle in W_L by setting $U = n.Y - S$ and we define $u=[U]\in H=H_2(W_L)$.
First, since the following diagram is commutative:

$$\begin{tikzcd}
H_2(W_L,V_L) \arrow[r, "\partial_*"] \arrow[d, "P"] & H_1(V_L) \arrow[d, "P"] \\
H^2(W_L) \arrow[r, "i^*"] & H^2(V_L),
\end{tikzcd}$$

we have $v\left([P(y)]\right) = P(x)$.
Second, $\frac{1}{n}u\in H\otimes Q$ belongs to the dual lattice $H^\sharp$: indeed, $P^{-1}\widehat{f}(u)=i_*(u)\in H_2(W_L,V_L)$ equals $n{\cdot}y$ so that $\widehat{f}(u)=n{\cdot}P(y)$. Moreover, we have then: $\widehat{f_{\mathbb{Q}}}\left(\frac{1}{n}u\right)|_H = P(y)$. Since, the map $G_f\to \mathrm{T}\ \mathrm{Coker}\widehat{f}$ featured by the short exact sequence (2.4) is defined by $[x]\mapsto \left[\widehat{f_{\mathbb{Q}}}(x)|_H\right]$, it sends $\left[\frac{1}{n}\cdot u\right]$ to $[P(y)]$.
Third, $\left[S\otimes\left[\frac{1}{n}\right]\right]\in H_2(V_L;\mathbb{Q}/\mathbb{Z})$ is well-defined and goes by B to x.
Finally, the canonical map $H_2(W_L;\mathbb{Q})\to H_2(W_L;\mathbb{Q}/\mathbb{Z})\simeq H_2(W_L)\otimes\mathbb{Q}/\mathbb{Z}$ sends $\frac{1}{n}u$ to $u\otimes\frac{1}{n} = \left[(n\cdot Y - S)\otimes\left[\frac{1}{n}\right]\right]$. Consequently, we get $\kappa\left(\left[\frac{1}{n}\cdot u\right]\right) = -\left[S\otimes\left[\frac{1}{n}\right]\right]$. Putting those four facts all together, we conclude the proof of the lemma. □

4.1.2. *Definition of* $\phi_{M,\sigma}$. Let $c\in \mathrm{Char}(f)$ represent $\psi^*(\sigma)\in \mathrm{Spin}^c(V_L)$ as explained in §3.4.2. To the bilinear lattice with characteristic form (H,f,c), we can associate (see §2.2) a quadratic function $\phi_{f,c}: G_f\to\mathbb{Q}/\mathbb{Z}$. Recall that $\phi_{f,c}$ only depends on the class $[c]$ of c in $\mathrm{Char}(f)/2\cdot \mathrm{Im}\widehat{f}$.

DEFINITION 4.3. *The quadratic function associated to the* Spin^c*-manifold* (M,σ) *is the following composition:*

$$\begin{tikzcd}
H_2\left(M;\mathbb{Q}/\mathbb{Z}\right) \arrow[r, "\psi_*", "\simeq"'] \arrow[rrr, dotted, bend right, "\phi_{M,\sigma}"'] & H_2\left(V_L;\mathbb{Q}/\mathbb{Z}\right) \arrow[r, "\kappa^{-1}", "\simeq"'] & G_f \arrow[r, "-\phi_{f,c}"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

PROOF OF THE WELL-DEFINITION OF $\phi_{M,\sigma}$. We have to verify that Definition 4.3 does not depend on the choice of the surgery presentation. Suppose that $\psi': V_{L'}\to M$ is another surgery presentation; quantities with a prime will be relative to this presentation. According to Kirby's theorem, there exists a path of Kirby's moves from L to L' inducing a diffeomorphism $h: V_L\to V_{L'}$ isotopic to $(\psi')^{-1}\circ\psi$;

we have to verify the commutativity of the diagram:

$$\begin{tikzcd}
H_2(V_L;\mathbb{Q}/\mathbb{Z}) \arrow[r, "\kappa^{-1}", "\simeq"'] \arrow[dd, "h_*", "\simeq"'] & \frac{H^\sharp}{H} \arrow[dr, "-\phi_{f,c}"] & \\
& & \mathbb{Q}/\mathbb{Z} \\
H_2(V_{L'};\mathbb{Q}/\mathbb{Z}) \arrow[r, "\kappa'^{-1}", "\simeq"'] & \frac{H'^\sharp}{H'} \arrow[ur, "-\phi_{f',c'}"'] &
\end{tikzcd}$$

where $c' \in \mathrm{Char}(f')$ is representative for $(\psi')^*(\sigma)$. Equivalently, define the isomorphism $t : G_f \to G_{f'}$ by:

$$\begin{tikzcd}
G_f \arrow[r, "\kappa", "\simeq"'] \arrow[d, dotted, "t"'] & H_2(V_L;\mathbb{Q}/\mathbb{Z}) \arrow[d, "h_*"', "\simeq"] \\
G_{f'} \arrow[r, "\kappa'", "\simeq"'] & H_2(V_{L'};\mathbb{Q}/\mathbb{Z});
\end{tikzcd}$$

we then have to verify that: $\phi_{f,c} = \phi_{f',c'} \circ t$. We can suppose that L and L' are related by one single Kirby's move.

If this Kirby's move is an orientation reversal or a handle sliding move, the diffeomorphism h is induced by a diffeomorphism $W_L \to W_{L'}$, inducing itself an isomorphism $T : H \to H'$ making thus f and f' commute. Note that the above defined t is induced by T. Also T induces its dual $T^* : (H')^* \to H^*$ which in turn induces a bijection $\mathrm{Char}(f') \to \mathrm{Char}(f)$. Note that c' can be choosen to be the inverse image of c by T^*. Then, for all $[x] \in H^\sharp/H$, we have:

$$\begin{aligned}
\phi_{f',c'}\left(t([x])\right) &= \phi_{f',c'}\left([T_{\mathbb{Q}}(x)]\right) \\
&= \tfrac{1}{2}\cdot\left(f'_{\mathbb{Q}}(T_{\mathbb{Q}}(x), T_{\mathbb{Q}}(x)) - c'_{\mathbb{Q}}(T_{\mathbb{Q}}(x))\right) \\
&= \tfrac{1}{2}\cdot\left(f_{\mathbb{Q}}(x,x) - c_{\mathbb{Q}}(x)\right) \\
&= \phi_{f,c}\left([x]\right)
\end{aligned}$$

Finally, if the Kirby's move is a stabilization, we then have $H' = H \oplus \mathbb{Z}$, $f' = f \oplus (\pm 1)$ and a formula for t is: $\forall [x] \in H^\sharp/H$, $t([x]) = [(x, 0)]$. As a $c' \in (H')^*$, we can take $c \oplus \mathrm{Id}_{\mathbb{Z}}$ (by Theorem 3.33). We then obtain immediately that $\phi_{f',c'}(t([x])) = \phi_{f,c}([x])$, thus concluding the proof of the well-definition of $\phi_{M,\sigma}$. □

EXAMPLE 4.4. For instance, suppose that M is a rational homology sphere and that L is *algebraically split*, that is, the linking matrix $B_L = (b_{ij})_{ij}$ is diagonal. Denote by m_i the meridian of L_i oriented so that: $\mathrm{lk}_{\mathbf{S}^3}(L_i, m_i) = +1$ and let $[m_i] \in H_1(M) = \mathrm{T}H_1(M) \simeq H_2(M;\mathbb{Q}/\mathbb{Z})$ be its homology class. Let also $s \in \mathcal{S}^c_L \simeq \mathrm{Char}(f)/2\cdot\mathrm{Im}\widehat{f}$ represent the Spin^c-structure σ. It can then be verified from the definitions that:

$$\phi_{M,\sigma}([m_i]) = -\frac{1}{2b_{ii}}(1 - s_i) \bmod 1. \tag{4.2}$$

DEFINITION 4.5. The *linking pairing* on M:

$$L_M : H_2(M;\mathbb{Q}/\mathbb{Z}) \times H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

is the composition of the torsion linking pairing $\lambda_M : \mathrm{T}H_1(M) \times \mathrm{T}H_1(M) \to \mathbb{Q}/\mathbb{Z}$ with $B \times B$.

The map L_M is thus a symmetric bilinear form, possibly degenerate with $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$ as radical.

LEMMA 4.6. *$\phi_{M,\sigma}$ is a quadratic function over L_M, that is:*

$$\forall x, y \in H_2(M;\mathbb{Q}/\mathbb{Z}), \quad \phi_{M,\sigma}(x+y) - \phi_{M,\sigma}(x) - \phi_{M,\sigma}(y) = L_M(x,y).$$

PROOF. It is well-known that the linking pairing λ_M can be calculated from the surgery presentation $\psi : V_L \to M$ as follows. On the one hand, the isomorphism $H^2(V_L) \to \text{Coker } \widehat{f}$ described in Remark 3.35, together with ψ and the Poincaré duality, induces an isomorphism $\mathrm{T}H_1(M) \to \mathrm{T}$ Coker $\widehat{f}$; on the other hand, we considered in §2.2 a symmetric nondegenerate bilinear form $\lambda_f : \mathrm{T} \text{ Coker } \widehat{f} \times \mathrm{T} \text{ Coker } \widehat{f} \to \mathbb{Q}/\mathbb{Z}$. Then, λ_M corresponds to $-\lambda_f$ *via* that isomorphism $\mathrm{T}H_1(M) \to \mathrm{T}$ Coker $\widehat{f}$. So, according to Lemma 4.2, the map L_M is determined by the following commutative diagram:

$$\begin{tikzcd}
H_2(M;\mathbb{Q}/\mathbb{Z}) \times H_2(M;\mathbb{Q}/\mathbb{Z}) \arrow[r, "L_M"] \arrow[d, "\psi_* \times \psi_*"'] & \mathbb{Q}/\mathbb{Z} \arrow[dd, equal] \\
H_2(V_L;\mathbb{Q}/\mathbb{Z}) \otimes H_2(V_L;\mathbb{Q}/\mathbb{Z}) & \\
G_f \otimes G_f \arrow[u, "\kappa \times \kappa"] \arrow[r, "-L_f"] & \mathbb{Q}/\mathbb{Z},
\end{tikzcd}$$

where the form L_f has been introduced in §2.2. The lemma then follows from the fact that $\phi_{f,c}$ is a quadratic function over L_f. □

Next, we investigate the propertie of the map:

$$\mathrm{Spin}^c(M) \xrightarrow{\phi_M} \mathrm{Quad}(L_M),$$

defined by $\sigma \mapsto \phi_{M,\sigma}$.

4.2. Properties of ϕ_M. In the sequel, we fix a connected closed oriented 3-manifold M. Next lemma says that, for any Spin^c-structure σ on M, the quadratic function $\phi_{M,\sigma}$ is determined on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$ by the Chern class $c(\sigma)$; note that if $x \in H_2(M)$, then $\langle c(\sigma), x\rangle \in \mathbb{Z}$ is even since the mod 2 reduction of $c(\sigma)$ is $w_2(M) = 0$.

LEMMA 4.7. *If $\sigma \in \mathrm{Spin}^c(M)$, then $\phi_{M,\sigma}$ is linear on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$, and we have:*

$$\forall x \otimes [r] \in H_2(M) \otimes \mathbb{Q}/\mathbb{Z}, \quad \phi_{M,\sigma}(x \otimes [r]) = \frac{\langle c(\sigma), x\rangle}{2} \cdot [r].$$

PROOF. The first statement follows from the fact that $\mathrm{Ker}\,\widehat{(L_M)} = H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$. As for the second statement, it suffices to show it when $M = V_L$; suppose that σ is represented by the characteristic form $c \in \mathrm{Char}(f)$ and that $x \in H_2(V_L)$ goes to y in $H = H_2(W_L)$. Then, $x \otimes [r]$ as an element of $H_2(V_L;\mathbb{Q}/\mathbb{Z})$ corresponds to $[r \cdot y]$ in $H^\sharp/H$. We then have: $\phi_{M,\sigma}(x \otimes [r]) = -\phi_{f,c}([r \cdot y]) = -\frac{1}{2}\left(r^2 f(y,y) - r \cdot c(y)\right)$ mod 1, but since $y \in \mathrm{Ker}\, \widehat{f}$, we obtain: $\phi_{M,\sigma}(x \otimes [r]) = \frac{1}{2} \cdot r \cdot c(y)$ mod 1, and according to Remark 3.35, $c(y) = \langle c(\sigma), x\rangle$. □

REMARK 4.8. If $\sigma \in \mathrm{T}\ \mathrm{Spin}^c(M)$, $c(\sigma)$ is torsion and Lemma 4.7 implies that $\phi_{M,\sigma}$ vanishes on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$ and so $\phi_{M,\sigma}$ factorizes to a quadratic function over λ_M. Then, in this particular case, our construction is easily seen to agree with that of [**De3**] and, up to a minus sign, with that of [**Gi**] (see Remark 2.6).

In particular, if σ is induced by a Spin-structure on M *via* the canonical map $\beta : \mathrm{Spin}(M) \to \mathrm{Spin}^c(M)$, then the factorization of $\phi_{M,\sigma}$ to $\mathrm{T}H_1(M)$ coincides with the quadratic form defined in [**LL**], [**MS**] or [**T2**]. In Chapter 2, this quadratic form is shown to play a basic role in the Goussarov-Habiro finite type invariants theory for manifolds with Spin-structures.

LEMMA 4.9. *If $\sigma \in \mathrm{Spin}^c(M)$, the homogeneity defect of the quadratic function $\phi_{M,\sigma}$:*

$$\phi_{M,\sigma} : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

is then defined by $x \mapsto \langle c(\sigma), x\rangle$.

PROOF. Again suppose that $M = V_L$ and that σ is represented by $c \in \mathrm{Char}(f)$. Take $x \in H_2(V_L;\mathbb{Q}/\mathbb{Z})$ represented by $y \in H^\sharp$, then $\phi_{M,\sigma}(x) - \phi_{M,\sigma}(-x) = -\phi_{f,c}([y]) + \phi_{f,c}(-[y]) = c_{\mathbb{Q}}(y) = \langle c(\sigma), x\rangle$ by Remark 3.35. □

Recall that $\mathrm{Spin}^c(M)$ is an affine space over $H^2(M)$ and that $\mathrm{Quad}(L_M)$ is an affine space over $\mathrm{Hom}\,(H_2(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$. We denote by:

$$H^2(M) \xrightarrow{\mu_M} \mathrm{Hom}\,(H_2(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$$

the homomorphism defined by $\mu_M(y) = \langle y, -\rangle$, and we denote by:

$$\mathrm{Hom}\,(H_2(M),\mathbb{Z}) \xrightarrow{j_M} \mathrm{Hom}\,(H_2(M)\otimes\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$$

the homomorphism given by $j_M(l) = l \otimes \mathrm{Id}_{\mathbb{Q}/\mathbb{Z}}$. Next results follow directly from Th. 2.9.

THEOREM 4.10. *The map $\phi_M : \mathrm{Spin}^c(M) \to \mathrm{Quad}(L_M)$ is an affine embedding over the group monomorphism μ_M. Furthermore,*

$$\mathrm{Coker}\ \phi_M \simeq \mathrm{Coker}\ j_M = \frac{\mathrm{Hom}\,(H_2(M)\otimes\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})}{j_M\,(\mathrm{Hom}(H_2(M),\mathbb{Z}))},$$

and given $q \in \mathrm{Quad}(L_M)$, q belongs to $\mathrm{Im}\ \phi_M$ if and only if $q|_{H_2(M)\otimes\mathbb{Q}/\mathbb{Z}}$ belongs to $\mathrm{Im}\ j_M$.

COROLLAIRE 4.1. *The map ϕ_M is bijective if and only if M is a rational homology sphere.*

The next paragraph provides an alternative, perhaps more appealing, description of the cokernel of the affine map ϕ_M (and shows that, in a sense, it is not big).

4.3. The cokernel of ϕ_M. In this paragraph only, $\mathbb{Z}_p$ denotes the ring of p-adic integers. For any prime p, there is a natural embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ and with respect to the p-adic topology, $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Let $\mathcal{P}$ denote the set of all primes. Equip $\widehat{\mathbb{Z}} = \prod_{p\in\mathcal{P}} \mathbb{Z}_p$ with the product (Tychonoff) topology. *Via* the diagonal embedding

$$\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}},$$

$\mathbb{Z}$ is again dense (but not closed) in $\widehat{\mathbb{Z}}$. Here is a context (well known to number theorists) in which this ring appears:

LEMMA 4.11. $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \simeq \widehat{\mathbb{Z}}$.

PROOF. For $p \in \mathcal{P}$, denote by A_p the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of all elements m/p^k (mod 1), $m \in \mathbb{Z}, k \in \mathbb{N}$. Then, $\mathbb{Q}/\mathbb{Z} \simeq \oplus_{p\in\mathcal{P}} A_p$. We have:

$$\begin{aligned}\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) &= \mathrm{Hom}\,(\oplus_{p\in\mathcal{P}} A_p,\mathbb{Q}/\mathbb{Z})\\ &= \prod_{p\in\mathcal{P}} \mathrm{Hom}\,(A_p,\mathbb{Q}/\mathbb{Z})\\ &= \prod_{p\in\mathcal{P}} \mathrm{Hom}\,(A_p,A_p)\end{aligned}$$

Now any endomorphism f of A_p is fully determined by the images of $\ldots$, $1/p^{k+1}$, $1/p^k$, $\ldots$ in A_p, which must be compatible in the obvious sense. We recover in this fashion the definition of $\mathbb{Z}_p$ as an inverse limit. Alternatively, A_p is itself a direct limit: $A_p = \lim\limits_{\rightarrow} \langle 1/p^k \rangle$, $k \in \mathbb{N}$, where $\langle 1/p^k \rangle$ is the additive subgroup generated by $1/p^k$ in A_p. So, $\operatorname{Hom}(A_p, A_p) = \operatorname{Hom}\left(\lim\limits_{\rightarrow} \langle 1/p^k \rangle, A_p\right) \simeq \lim\limits_{\leftarrow} \operatorname{Hom}\left(\langle 1/p^k \rangle, A_p\right) \simeq \lim\limits_{\leftarrow} \operatorname{Hom}\left(\langle 1/p^k \rangle, \mathbb{Q}/\mathbb{Z}\right) \simeq \mathbb{Z}_p$. □

Therefore, from Lemma 4.11, the embedding j_M involved in Theorem 4.10 is interpreted as the diagonal embedding $\delta_M : H_2(M) \hookrightarrow H_2(M) \otimes \widehat{\mathbb{Z}}$. Hence, we have:

$$\text{Coker } \phi_M \simeq \text{Coker } \delta_M = \frac{H_2(M) \otimes \widehat{\mathbb{Z}}}{\delta_M\left(H_2(M)\right)}.$$

4.4. An intrinsic definition for $\phi_{M,\sigma}$. Let M be a closed connected oriented 3-manifold and let σ be a Spin^c-structure on M. The goal of this paragraph is to give an intrinsic definition for the quadratic function $\phi_{M,\sigma}$, *i.e.* we want it not to be dependent on the choice of a 4-dimensional bordism.

Here is the idea. As it is easily verified from Lemma 4.9, we have for any $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$:

$$2 \cdot \phi_{M,\sigma}(x) = L_M(x,x) + \langle c(\sigma), x \rangle \in \mathbb{Q}/\mathbb{Z}.$$

If $y \in \mathbb{Q}/\mathbb{Z}$, let us denote by $\frac{1}{2} \cdot y$ the set of elements z of $\mathbb{Q}/\mathbb{Z}$ such that $z + z = y$. Then, we are going to pick, in a *correlative* way, a $z_1 \in \frac{1}{2} \cdot L_M(x,x)$ and a $z_2 \in \frac{1}{2} \cdot \langle c(\sigma), x \rangle$ such that $\phi_{M,\sigma}(x) = z_1 + z_2$.

Definition 4.12. An element x of $H_2(M;\mathbb{Q}/\mathbb{Z})$ is said here to be *special* if it can be written as $\left[S \otimes \left[\frac{1}{n}\right]\right]$, where $n \in \mathbb{N}^\times$ and where S is an oriented immersed surface in M with boundary $n \cdot K$, a bunch of n parallel copies of a knot K in M.

Lemma 4.13. *Every element x of $H_2(M;\mathbb{Q}/\mathbb{Z})$ is special modulo $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$.*

Proof. The class $B(x) \in H_1(M)$ being torsion, there exists some $n \in \mathbb{N}^\times$ such that $n \cdot B(x) = 0$. Let an oriented knot K realize $B(x)$, and let S be an immersed surface with boundary $n \cdot K$. Then, $S \otimes \left[\frac{1}{n}\right]$ is a 2-cycle with coefficients in $\mathbb{Q}/\mathbb{Z}$ whose image by B is $B(x)$. The lemma then follows from the fact $\operatorname{Ker}(B) = H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$. □

Let $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$. Writing x as $x' + x''$, where x' is special and x'' belongs to $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$, we obtain:

$$\phi_{M,\sigma}(x) = \phi_{M,\sigma}(x') + \phi_{M,\sigma}(x'').$$

Lemma 4.7 gives an intrinsic formula for $\phi_{M,\sigma}(x'')$ (then: $\phi_{M,\sigma}(x'') \in \frac{1}{2} \cdot \langle c(\sigma), x'' \rangle$), so that we can restrict ourselves to the special case.

In the sequel, we take a special element x of $H_2(M;\mathbb{Q}/\mathbb{Z})$ written as $x = \left[S \otimes \left[\frac{1}{n}\right]\right]$, as detailed in Definition 4.12. We now give the stepwise recipe for a 3-dimensional formula for $\phi_{M,\sigma}(x)$.

- *Step 1.* Choose a nonsingular tangent vector field v to M representing σ and which is transverse to K (we claim that it is possible to find such v).
- *Step 2.* Let V be a sufficiently small regular neighborhood of K in M and let K_v be the parallel of K, lying on ∂V, obtained by pushing K along the trajectories of v. Isotope S so that $\partial S \subset \operatorname{int}(V)$ and S is in transverse position with K_v.

- *Step 3.* Define a Spin-structure α_v on $\partial\left(M \setminus \mathrm{int}\left(V\right)\right)$ by requiring its Johnson quadratic form q_{α_v} (§3.2.5) to be such that:

$$q_{\alpha_v}\left(\text{meridian of } K\right) = 0 \quad \text{and} \quad q_{\alpha_v}(K_v) = 1.$$

- *Step 4.* Then, v represents a Spin^c-structure σ_v on $M \setminus \mathrm{int}\left(V\right)$ relative to the Spin-structure α_v (we claim this). Let

$$c(\sigma_v) \in H^2\left(M \setminus \mathrm{int}\left(V\right), \partial\left(M \setminus \mathrm{int}\left(V\right)\right)\right)$$

be the relative Chern class of σ_v.

PROPOSITION 4.14. *From the previous recipe, we get:*

$$\phi_{M,\sigma}(x) = \underbrace{\left[\frac{1}{2n} \cdot K_v \bullet S\right]}_{\in \frac{1}{2}\cdot L_M(x,x)} + \underbrace{\left[\frac{1}{2n} \cdot \langle c(\sigma_v), [S \cap (M \setminus \mathrm{int}\,(V))]\rangle + \frac{1}{2}\right]}_{\in \frac{1}{2}\cdot\langle c(\sigma), x\rangle} \in \mathbb{Q}/\mathbb{Z}. \tag{4.3}$$

REMARK 4.15. In [**LW**], Looijenga and Wahl associate a quadratic function over λ_M, to each pair $(M, \mathcal{J})$, where

(1) M is a closed connected oriented 3-manifold,
(2) $\mathcal{J}$ is a homotopy class of complex structures on $\epsilon^1 \oplus T_M$ whose first Chern class is torsion

(here ϵ^1 denotes the trivial real one-dimensional vector bundle over M, so that $\epsilon^1 \oplus T_M$ is an oriented real 4-dimensional vector bundle).

Suppose instead now that M comes equipped with a nonsingular section v of T_M representing a *torsion* Spin^c-structure σ, and endow it with an arbitrary Riemannian metric. Then $\epsilon^1 \oplus T_M = \left(\epsilon^1 \oplus \langle v\rangle\right) \oplus \langle v\rangle^\perp$ is written as the sum of two oriented real 2-dimensional vector bundles. So *via* the inclusion $U(1) \times U(1) \hookrightarrow U(2)$, v defines a complex structure J_v on $\epsilon^1 \oplus T_M$. The first Chern class of J_v is that of $\langle v\rangle^\perp$ so equals $c(\sigma)$ and then is torsion. The quadratic function defined by J_v coincides with $\phi_{M,\sigma}$ (compare formula (3.4.1) in [**LW**] with our formula (4.3)).

PROOF OF PROP. 4.14. First of all, we have to justify that the provided recipe can actually be carried out.
We begin by proving the claim of Step 1. Let v be an arbitrary nonsingular tangent vector field to M representing σ. It suffices to prove the following claim.

CLAIM 4.16. Let w be an arbitrary nonsingular tangent vector field to M defined on K. Then, v can be homotoped so as to coincide with w on K.

PROOF. Choose a tubular neighborhood W of K, plus an identification $W = (2\mathbf{D}^2) \times \mathbf{S}^1$ such that K corresponds to $0 \times \mathbf{S}^1$. We denote by (e_1, e_2) the standard basis of $\mathbb{R}^2 \supset 2\mathbf{D}^2$. We define $\pi : W \to K$ to be the projection on the core. The solid torus W is parametrized by the cylindric coordinates:

$$\left(\left(r \in [0,2], \theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}\right), \phi \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}\right).$$

If $p, q \in W$ are such that $\phi(p) = \phi(q)$ (*i.e.* they belong to the same meridional disk $2\mathbf{D}^2 \times *$), we define the transport map $t_{p,q} : T_pW \to T_qW$ as the unique linear map fixing the basis $\left(e_1, e_2, \frac{\partial}{\partial\phi}\right)$. The vector field v can be homotoped to a vector field which is constant on each meridional disk $\mathbf{D}^2 \times *$; this is accomplished by the homotopy $\left(v^{(t)}\right)_{t\in[0,1]}$ given at time t and point $p \in W$ by:

$$v_p^{(t)} = \begin{cases} t_{\pi(p),p}\left(v_{\pi(p)}\right) & \text{if } r(p) \in [0,t] \\ t_{q(p,t),p}\left(v_{q(p,t)}\right) & \text{if } r(p) \in [t,2], \text{ with } q(p,t) = \left(\frac{r(p)-t}{1-t/2}, \theta(p), \phi(p)\right) \end{cases}$$

and at time t and point $p \notin W$ by: $v_p^{(t)} = v_p$. After such a deformation, the vector field v satisfies: $\forall p \in \mathbf{D}^2 \times \mathbf{S}^1$, $t_{p,\pi(p)}(v_p) = v_{\pi(p)}$. Now, since $\pi_1(\mathbf{S}^2)$ is trivial, $v|_K$ and w have to be homotopic: let $\left(w^{(t)}\right)_{t\in[0,1]}$ be such a homotopy, beginning at $w^{(0)} = v|_K$ and ending at $w^{(1)} = w$. The homotopy given by:

$$v_p^{(t)} = \begin{cases} t_{\pi(p),p}\left(w_{\pi(p)}^{(t-r(p))}\right) & \text{if } r(p) \in [0,t] \\ v_p & \text{if } r(p) \in [t,2] \end{cases}$$

if $p \in W$ and by $v_p^{(t)} = v_p$ if $p \notin W$, allows us to suppose that v coincides with w on K. □

Let now V be a regular neighborhood of K in M, plus an identification $V = \mathbf{D}^2 \times \mathbf{S}^1$ such that K corresponds to $0 \times \mathbf{S}^1$ and such that $v|_V$ corresponds to e_1 (recall that (e_1, e_2) denotes the standard basis of $\mathbb{R}^2 \supset \mathbf{D}^2$). We apply steps 2 and 3 (note that K_v then corresponds to $1 \times \mathbf{S}^1$) and we now prove the claim of Step 4. Let $\tau_v \in \mathrm{Spin}(V)$ correspond to the parallelization $\left(e_1, e_2, \frac{\partial}{\partial\theta}\right)$ of V. Since $(\tau_v|_{\partial V})|_{1\times\mathbf{S}^1}$ is the non-bouding Spin-structure and since $(\tau_v|_{\partial V})|_{\partial\mathbf{D}^2\times 1}$ spin bounds, we have $\tau_v|_{\partial V} = -\alpha_v$, *i.e.* $\tau_v \in \mathrm{Spin}(V, -\alpha_v)$ with the notation of Remark 3.13. Thus, v being e_1 on V, it well-defines a Spin^c-structure σ_v on $M \setminus \mathrm{int}\,(V)$ relative to α_v, as claimed in Step 4. For futher use, note that σ is the gluing $\sigma_v \cup \beta(\tau_v)$, where $\beta : \mathrm{Spin}(V, -\alpha_v) \to \mathrm{Spin}^c(V, -\alpha_v)$ has been defined in Remark 3.13.

Set $z_1 = [1/2n \cdot K_v \bullet S] \in \mathbb{Q}/\mathbb{Z}$ and $z_2 = [1/2n \cdot \langle c(\sigma_v), [S'] \rangle + 1/2] \in \mathbb{Q}/\mathbb{Z}$, where $S' = S \cap (M \setminus \mathrm{int}\,(V))$. We then have: $2\cdot z_1 = [1/n \cdot K_v \bullet S] = [\mathrm{lk}_M\,(K, K_v)] = [\lambda_M\,(B(x), B(x))] = L_M(x,x)$. Moreover:

$$\begin{aligned} 2 \cdot z_2 &= [1/n \cdot \langle c(\sigma_v), [S'] \rangle] \\ &= \left[1/n \cdot P^{-1}\left(c(\sigma_v)\right) \bullet [S']\right] \text{ (intersection in } M \setminus \mathrm{int}(V)) \\ &= P^{-1}\left(c(\sigma)\right) \bullet x \text{ (intersection in } M) \\ &= \langle c(\sigma), x \rangle \end{aligned}$$

where the third equality follows from the facts that $x = \left[S \otimes \left[\frac{1}{n}\right]\right]$, $P^{-1}\left(c(\sigma)\right) = i_* P^{-1}\left(c(\sigma_v)\right) + i_* P^{-1}\left(c(\beta(\tau_v))\right) \in H_1(M)$ (since $\sigma = \sigma_v \cup \beta(\tau_v)$) and $c(\beta(\tau_v)) = 0$ (by Remark 3.25).

We now prove formula (4.3), that is $\phi_{M,\sigma}(x) = z_1 + z_2$. We begin by defining a particular surgery presentation of M. Construct a 3-manifold M' from M by doing surgery along the framed knot $(K, (e_1, e_2))$. Pick a surgery presentation $V_{L'}$ of M'; up to isotopy, the dual knot K' of K in M' lies in $\mathbf{S}^3 \setminus L'$. We then find a surgery presentation V_L of M by setting L to be L' union K' with the appropriate framing. This surgery presentation of M has the following advantage: K bounds in the trace W_L of the surgery a disk D whose normal bundle is trivialized by some extensions of the trivialization (e_1, e_2) of the normal bundle of K in M.

For such a surgery presentation, we use the notations fixed in §3.4. In particular, $H = H_2(W_L)$ and $f : H \times H \to \mathbb{Z}$ is the intersection pairing in W_L. We define the 2-cycle $U = n \cdot D - S$ where $n \cdot D$ is a bunch of n parallel copies of the disk D with boundary $n \cdot K$; we also define $u = [U] \in H$. Then $\frac{1}{n} \cdot u$ belongs to $H^\sharp$ and the isomorphism $\kappa : \frac{H^\sharp}{H} \to H_2(M; \mathbb{Q}/\mathbb{Z})$ sends $\left[\frac{1}{n} u\right]$ to $-x = -\left[S \otimes \left[\frac{1}{n}\right]\right]$ (see the proof of Lemma 4.2). So, by definition:

$$\phi_{M,\sigma}(x) = -\phi_{f,c}\left(-\left[\frac{1}{n} \cdot u\right]\right) = -\frac{1}{2}\left(\frac{1}{n^2} f(u,u) + \frac{1}{n} c(u)\right),$$

where c is a characteristic form representative for σ.

Let us calculate the quantity $f(u,u)$. The 2-cycle U is a representant of u, a second one is $U' = n\cdot D' + n\cdot A - S$ where D' is a push-off of D by the extension of $e_1 = v|_V$ such that $\partial D' = K_v$, and where A is the annulus of an isotopy from $-K_v$ to K in V (*e.g.* $A = -[0,1]\times \mathbf{S}^1$ in $V = \mathbf{D}^2 \times \mathbf{S}^1$). By adding a collar to W_L and stretching the top of U', we can make U in transverse position with U' (see Figure 4.1). So, $f(u,u) = U \bullet U' = -nS \bullet K_v$ where the first intersection is calculated in

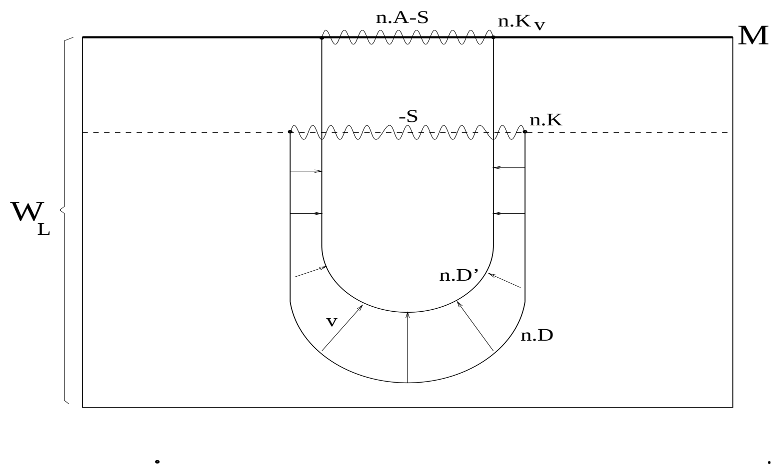

Figure 4.1: Two representants of u in transverse position

W_L and the second in M; we are led to:

$$\phi_{M,\sigma}(x) = \frac{1}{2n} S \bullet K_v - \frac{1}{2n} c(u). \tag{4.4}$$

We are now interested in the quantity $c(u)$. Let $\tilde{\sigma}$ be an extension of σ to the manifold W_L and let ξ be the isomorphism class of principal $U(1)$-bundles on W_L defined by $\tilde{\sigma}$; then c can be choosen to be $c_1(\xi)$. Let p be a representant of ξ and let tr be a trivialization of p on ∂V. Decompose the singular surface U' as $U' = U'_1 \cup U'_2 \cup U'_3$, where $U'_1 = n\cdot D'$, $U'_2 = n\cdot A \cup (-S\cap V)$ and $U'_3 = -S'$. By desingularizing U' so as to be reduced to a calculus of obstructions in an oriented manifold, we obtain:

$$c(u) = \langle c_1\left(p|_{U'}\right), [U']\rangle = \sum_{i=1}^{3} \left\langle c_1\left(p|_{U'_i}, \mathrm{tr}|_{\partial U'_i}\right), [U'_i]\right\rangle \tag{4.5}$$

where $c_1\left(p|_{U'_i}, \mathrm{tr}|_{\partial U'_i}\right) \in H^2\left(U'_i, \partial U'_i\right)$ is the obstruction to extend the trivialization $\mathrm{tr}|_{\partial U'_i}$ of $p|_{U'_i}$ on $\partial U'_i$ to the whole of U'_i. Let $T \subset W_L$ be the solide torus such that $M' = M \setminus \mathrm{int}(V) \cup T$. Pick a relative Spin^c-structure $\sigma_1 \in \mathrm{Spin}^c(T, \alpha_v)$ such that $\sigma_v \cup \sigma_1 = \tilde{\sigma}|_{M'}$. Then, for some appropriate choices of p in the class ξ and tr, we have:

$$\begin{aligned} c_1\left(p|_T, \mathrm{tr}\right) &= c(\sigma_1) \in H^2(T, \partial T),\\ c_1\left(p|_V, \mathrm{tr}\right) &= c(\beta(\tau_v)) \in H^2(V, \partial V),\\ c_1\left(p|_{M\setminus \mathrm{int}(V)}, \mathrm{tr}\right) &= c(\sigma_v) \in H^2\left(M\setminus \mathrm{int}(V), \partial\left(M\setminus \mathrm{int}(V)\right)\right). \end{aligned}$$

Then, equation (4.5) becomes:

$$c(u) = n\cdot \langle c(\sigma_1), [D']\rangle + \langle c(\beta(\tau_v)), [U'_2]\rangle - \langle c(\sigma_v), [S']\rangle \in \mathbb{Z}.$$

From the fact that $c(\beta(\tau_v)) = 0$, we then deduce that:

$$\frac{1}{2n} \cdot c(u) = -\frac{1}{2n} \cdot \langle c(\sigma_v), [S'] \rangle + \frac{1}{2} \cdot \langle c(\sigma_1), [D'] \rangle.$$

Then, showing that $\langle c(\sigma_1), [D'] \rangle$ is an odd integer together with (4.4) will end the proof of the proposition. Since $\langle c(\sigma_1), [D'] \rangle = q_{\alpha_v}(\partial[D']) = q_{\alpha_v}(K_v) = 1 \bmod 2$ (by Lemma 3.26), we are done. □

4.5. The maximal Abelian Reidemeister-Turaev torsion mod 1. Let X be a finite connected CW-complex with zero Euler characteristic and set $H = H_1(X)$, which is denoted multiplicatively. We denote by $Q(H)$ the ring of fractions of the ring $\mathbb{Z}[H]$. Turaev introduced in [**T1**] the *maximal Abelian Reidemeister torsion* of the complex X: denoted by $\tau(X)$, it belongs to $Q(H)$ and is defined up to multiplication by an element of $\pm H \subset Q(H)$. Moreover, it is shown in [**T4**] that this indeterminacy can be disposed of by specifying two further structures, called homology orientations and combinatorial Euler structures.

Suppose now that X is a connected closed oriented 3-manifold M. On the one hand, the orientation of M induces a canonical homology orientation; on the other hand, the combinatorial Euler structures on M are in canonical correspondance with the geometric Euler structures on M (§3.2.3) and according to Lemma 3.15, so they are with the Spin^c-structures on M. Therefore, if (M, σ) is any connected closed Spin^c-manifold of dimension 3, one can define its *maximal Abelian Reidemeister-Turaev torsion*: $\tau(M, \sigma) \in Q(H)$, where $H = H_1(M)$. It has the property that:

$$\forall h \in H_1(M),\ h \cdot \tau(M, \sigma) = \tau(M, P(h) \cdot \sigma) \in Q(H). \tag{4.6}$$

Suppose now that M is a rational homology sphere. Then, H is finite and it happens that $Q(H) = \mathbb{Q}[H]$. Let us now recall from [**T9**, Chap. X, §2.3] how, in this particular case, the torsion $\tau(M, \sigma)$ determines a quadratic function. Define a function $\tau_\sigma : H \to \mathbb{Q}$ by putting:

$$\tau(M, \sigma) = \sum_{h \in H} \tau_\sigma(h) \cdot h \in \mathbb{Q}[H].$$

The modulo 1 reduction of this function satisfies:

$$\forall h_1, h_2 \in H, \quad \tau_\sigma(h_1 h_2) - \tau_\sigma(h_1) - \tau_\sigma(h_2) + \tau_\sigma(1) = -\lambda_M(h_1, h_2) \mod 1, \tag{4.7}$$

where $\lambda_M : H \times H \to \mathbb{Q}/\mathbb{Z}$ is the torsion linking pairing of M. It immediately follows that:

$$\forall h \in H, \quad \tau_\sigma(h) = c_\sigma - q_{M,\sigma}\left(h^{-1}\right) \mod 1,$$

where $c_\sigma = \tau_\sigma(1) \in \mathbb{Q}/\mathbb{Z}$ is a constant and where $q_{M,\sigma}$ is a quadratic function over the linking pairing λ_M. It is easily seen from (4.6) and (4.7) that:

$$\forall h \in H, \quad q_{M, P(h)\cdot\sigma} = q_{M,\sigma} + \lambda_M(h, -). \tag{4.8}$$

Since M is a rational homology sphere, the Bockstein homomorphism B is then an isomorphism $H_2(M; \mathbb{Q}/\mathbb{Z}) \to H_1(M)$, so that that $\phi_{M,\sigma}$ can be regarded as a quadratic function $\phi_{M,\sigma} : H \to \mathbb{Q}/\mathbb{Z}$ over λ_M. Moreover, as follows from Theorem 4.10, we have:

$$\forall h \in H, \quad \phi_{M, P(h)\cdot\sigma} = \phi_{M,\sigma} + \lambda_M(h, -). \tag{4.9}$$

In fact, equations (4.8) and (4.9) are equivalent according to the following.

THEOREM 4.17. *For any rational homology 3-sphere M and any $\sigma \in \mathrm{Spin}^c(M)$, we have $q_{M,\sigma} = \phi_{M,\sigma}$.*

It follows in particular that the quadratic function $\phi_{M,\sigma}$ is determined by the maximal Abelian Reidemeister-Turaev torsion of (M, σ).

PROOF OF THEOREM 4.17. The technical difficulty lies in the computation of $q_{M,\sigma}$ from the torsion τ_σ. Fortunately, τ_σ can be computed from a surgery presentation of M and certain combinatorial data equivalent to Spin^c-structures, called *charges*.

We begin by recalling the definition of charges introduced in [**T8**]. We shall use notation of §3.4. Let $L \subset \mathbf{S}^3$ be an ordered oriented and framed n-component link. Surgery along L yields a closed oriented connected 3-manifold V_L. The manifold V_L is the union of the exterior E of L in $\mathbf{S}^3$ and the disjoint union of reglued solid tori Z_j, one for each component L_j of L. A solid torus Z is said to be *directed* when its core is oriented. We direct the solid torus Z_j in the following way: we denote by $m_j \subset E$ the meridian of L_j which is oriented so that $\mathrm{lk}_{\mathbf{S}^3}(m_j, L_j) = +1$, and we ask the oriented core of Z_j to be isotopic in V_L to m_j. A directed solid torus Z has a *distinguished* Spin^c-structure relative to the canonical (see Remark 3.12) spin structure σ_0 on ∂Z: this is the one whose Chern class is Poincaré dual to opposite of the oriented core of Z. Then, by gluing any Spin^c-structure on E relative to $\dot\cup\sigma_0$ (by means of Lemma 3.28) to the distinguished relative Spin^c-structure on the directed solid tori Z_j, we define a map $g : \mathrm{Spin}^c(E, \dot\cup\sigma_0) \to \mathrm{Spin}^c(V_L)$, which is affine, *via* the isomorphisms given by Poincaré dualities, over the natural map $H_1(E) \to H_1(V_L)$. Define the set $\mathcal{C}_L$ of *charges* on L by

$$\mathcal{C}_L = \left\{ k \in \mathbb{Z}^n \ : \ k_j = 1 + \sum_{\substack{1\leq i\leq n\\ i\neq j}} \mathrm{lk}_{\mathbf{S}^3}(L_i, L_j) \bmod 2 \right\}.$$

A canonical bijection between $\mathrm{Spin}^c(E, \dot\cup\sigma_0)$ and $\mathcal{C}_L$ can be defined in the following way: take $\sigma \in \mathrm{Spin}^c(E, \dot\cup\sigma_0)$, calculate $P^{-1}c(\sigma) \in H_1(E)$ and identify $H_1(E)$ with $\mathbb{Z}^n$ taking the meridians $([m_1], \ldots, [m_n])$ as a basis; it can then be derived from Lemma 3.26 that the multi-integer you get is actually a charge on L. Thus, since g is surjective and since $\mathrm{Ker}\,(H_1(E) \to H_1(V_L))$ is generated by the n characteristic curves of the surgery, it follows that the map g induces a bijection:

$$\frac{\mathcal{C}_L}{2 \cdot \mathrm{Im}\ B_L} \to \mathrm{Spin}^c(V_L)$$

where $B_L = (b_{ij})_{1\leq i,j\leq n}$ is the linking matrix of L in $\mathbf{S}^3$.

We now compare this combinatorial description of $\mathrm{Spin}^c(V_L)$ to our description described in §3.4.2 by showing that:

CLAIM 4.18. If $\sigma \in \mathrm{Spin}^c(V_L)$ corresponds to $[k] \in \mathcal{C}_L/2 \cdot \mathrm{Im}\ B_L$, then σ corresponds to $[s] \in \mathcal{S}^c_L$, where:

$$\forall j \in \{1, \ldots, n\}, \quad s_j = 1 - k_j + \sum_{i=1}^{n} b_{ij} \in \mathbb{Z}. \tag{4.10}$$

PROOF OF THE CLAIM 4.18. We denote by σ_2 the distinguished relative Spin^c-structure in $\mathrm{Spin}^c(\dot\cup Z_i, \dot\cup\sigma_0)$. Let also $\sigma_1 \in \mathrm{Spin}^c(E, \dot\cup\sigma_0)$ be such that: $\sigma = \sigma_1 \cup \sigma_2 \in \mathrm{Spin}^c(V_L)$. Pick an extension $\tilde\sigma$ of σ to W_L and let ξ be the isomorphism class of $U(1)$-principal bundles defined by $\tilde\sigma \in \mathrm{Spin}^c(W_L)$. On the one hand, the first Chern class $c_1(\xi)$ of ξ, when expressed in the preferred basis $([S_j]^*)_{j=1}^n$ of $H^2(W_L) \simeq \mathrm{Hom}\,(H_2(W_L), \mathbb{Z})$ (see §3.4), gives a multi-integer $s \in \mathbb{Z}^n$; then, $[s] \in \mathcal{S}^c_L$ corresponds to σ. On the other hand, as explained above, the Poincaré dual to the relative Chern class of $\sigma_1 \in \mathrm{Spin}^c(E, \dot\cup\sigma_0)$ when expressed in the preferred basis

$([m_j])_{j=1}^n$ of $H_1(E)$ gives a multi-integer $k \in \mathbb{Z}^n$; then, $[k] \in \mathcal{C}_L/2 \cdot \mathrm{Im}\ B_L$ corresponds to σ. Thus, proving that these specific k and s verify (4.10) modulo $2{\cdot}\mathrm{Im}\ B_L$ will be enough.

In the sequel we denote by $\mathbf{S}^3_\varepsilon$ a collar push-off of $\mathbf{S}^3 = \partial \mathbf{D}^4$ in the interior of $\mathbf{D}^4$. The surface S_j can be decomposed (up to isotopy) in W_L to:

$$S_j = D_j \cup A_j \cup \left(-\Sigma_j^{cut}\right)_\varepsilon \cup \bigcup_l \left(-R_{jl}\right)_\varepsilon ,$$

where the subsurfaces are given as follows (see Figure 4.2):

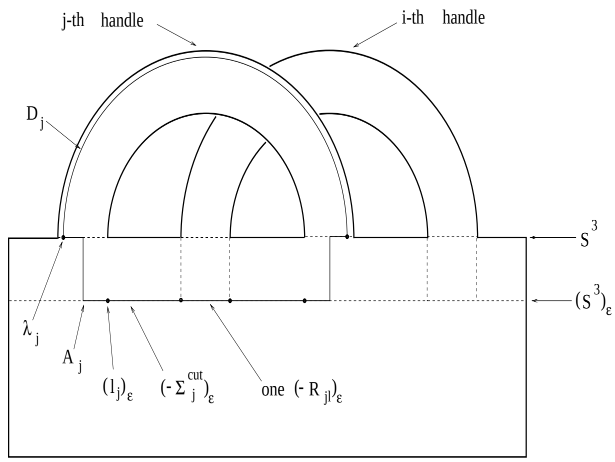

Figure 4.2: A decomposition of the surface S_j

- D_j is a meridian disc of Z_j such that ∂D_j is the characteristic curve λ_j of the j-th surgery;
- A_j is the annulus of an isotopy of $-\lambda_j$ to L_j, union the annulus of an isotopy of $-L_j$ to $(L_j)_\varepsilon$, union the annulus of an isotopy of $(-L_j)_\varepsilon$ to $(l_j)_\varepsilon$ where l_j denotes the preferred parallel of L_j in $\mathbf{S}^3$ (*i.e.* $\mathrm{lk}_{\mathbf{S}^3}(l_j, L_j) = 0$);
- Σ_j is a Seifert surface for l_j in $\mathbf{S}^3$ disjoint from L_j and in transverse position with the L_i $(i \neq j)$. For each intersection point x_l between Σ_j and a L_i, remove a small disc R_{jl} so that $\Sigma_j = \Sigma_j^{cut} \cup \bigcup_l R_{jl}$.

By definition of s, we have $s_j = \langle c_1(\xi), [S_j] \rangle = \langle c_1(p|_{S_j}), [S_j] \rangle$ where p is representative for ξ and where $c_1(p|_{S_j}) \in H^2(S_j)$ is the obstruction to trivialize p over S_j. So $P^{-1}c_1(p|_{S_j}) = s_j \cdot [\mathrm{pt}] \in H_0(S_j)$. Let tr be a trivialization of p on ∂E and let tr_ε be the corresponding trivialization of p on ∂E_ε. An argument similar to that used in the proof of Lemma 3.28 (calculus of obstructions in oriented manifolds) leads to:

$$\begin{aligned} (4.11) \qquad H_0(S_j) \ni P^{-1}c_1(p|_{S_j}) \quad = \quad & i_* P^{-1} c_1 \left(p|_{D_j}, \mathrm{tr}|_{\lambda_j}\right) \\ & + i_* P^{-1} c_1 \left(p|_{A_j}, \mathrm{tr}|_{-\lambda_j} \cup \mathrm{tr}_\varepsilon|_{(l_j)_\varepsilon}\right) \\ & - i_* P^{-1} c_1 \left(p|_{\left(\Sigma_j^{cut}\right)_\varepsilon}, \mathrm{tr}_\varepsilon|_{\left(\partial\Sigma_j^{cut}\right)_\varepsilon}\right) \\ & - \sum_l i_* P^{-1} c_1 \left(p|_{(R_{jl})_\varepsilon}, \mathrm{tr}_\varepsilon|_{(\partial R_{jl})_\varepsilon}\right), \end{aligned}$$

where P denotes the appropriate Poincaré duality isomorphism (of D_j, A_j, Σ_j^{cut} and R_{jl}). For an appropriate choice of p in the class ξ and for an appropriate choice of tr, we have:

$$\begin{aligned} c_1\left(p|_E, \mathrm{tr}\right) &= c(\sigma_1) \in H^2(E, \partial E) \\ c_1\left(p|_{\dot\cup Z_j}, \mathrm{tr}\right) &= c(\sigma_2) \in H^2\left(\dot\cup Z_j, \dot\cup \partial Z_j\right) \\ c_1\left(p|_{\mathrm{N}(L)}, \mathrm{tr}\right) &= c(\sigma_3) \in H^2\left(\mathrm{N}(L), \partial \mathrm{N}(L)\right), \end{aligned}$$

where, in this last requirement, $\mathrm{N}(L)$ is a tubular neighborhood of L in $\mathbf{S}^3$ and σ_3 is an arbitrary element of $\mathrm{Spin}^c\left(\mathrm{N}(L), \dot\cup\sigma_0\right)$. For such choices, we now compute separately each term of the right hand side of (4.11).

(1) The first term is of the form $d_j \cdot [\mathrm{pt}]$. Here:
$$d_j = \langle c(\sigma_2), [D_j]\rangle = -\left(\text{oriented core of } Z_j\right) \bullet [D_j] = +1,$$
where the intersection is taken in Z_j (note that $Z_j = \left(\mathbf{D}^2 \times \mathbf{S}^1\right)_j$ if we denote by $\left(\mathbf{D}^2 \times \mathbf{D}^2\right)_j$ the 2-handle of W_L corresponding to L_j, and be careful of the fact that the above specified oriented core of Z_j is $-\left(0 \times \mathbf{S}^1\right)_j$)

(2) The second term is of the form $a_j \cdot [\mathrm{pt}]$. Here: $a_j = \langle c(\sigma_3), [A_j]\rangle$ where A_j is regarded as a relative 2-cycle in $(\mathrm{N}(L), \partial\mathrm{N}(L))$ once the collar has been squeezed. Since ∂A_j is $-\lambda_j \cup l_j$, $[A_j]$ is $-b_{jj}$ times the class of the meridian disc of L_j (oriented so that its oriented boundary is m_j) in $H_2(\mathrm{N}(L), \partial\mathrm{N}(L))$. Then, $a_j = -b_{jj} \cdot \rho_j$ where:
$$\rho_j = \langle c(\sigma_3), [\text{meridian disc of } L_j]\rangle \in \mathbb{Z}.$$
Note that: $\rho_j = q_{\sigma_0}([m_j]) = 1 \bmod 2$ (Johnson's quadratic function, see §3.2.5).

(3) The third term is $-g_j \cdot [\mathrm{pt}]$ where $g_j = \langle c(\sigma_1), [\Sigma_j^{cut}]\rangle = \left(P^{-1}c(\sigma_1)\right) \bullet [\Sigma_j^{cut}] = \left(\sum_i k_i [m_i]\right) \bullet [\Sigma_j^{cut}] = \sum_i k_i \delta_{ij} = k_j$ where the intersection is taken in E.

(4) The fourth term is given by $-\sum_l r_{jl} \cdot [\mathrm{pt}]$. Here $r_{jl} = \langle c(\sigma_3), [R_{jl}]\rangle$. For each index l, denote by $i(l)$ the integer i such that x_l is an intersection point of Σ_j with L_i, and denote by $\epsilon(l)$ the sign of the intersection point x_l. Then, from the definition of ρ_i (given for the second term), we have: $r_{jl} = \epsilon(l) \cdot \rho_{i(l)}$. We then get:
$$\sum_l r_{jl} = \sum_{\substack{i=1 \\ i \neq j}}^n b_{ij} \rho_i.$$

Finally, we obtain:

$$\begin{aligned} s_j &= 1 - b_{jj}\rho_j - k_j - \sum_{\substack{i=1 \\ i \neq j}}^n b_{ij}\rho_i \\ &= \left(1 - k_j + \sum_{i=1}^n b_{ij}\right) - \sum_{i=1}^n b_{ij}\left(\rho_i + 1\right), \end{aligned}$$

the claim then follows from the fact that $\rho_i = 1 \bmod 2$ for all $i = 1, \ldots, n$. □

We are now able to prove the theorem. Assume first that M is obtained by surgery along an algebraically split link L, and that σ is represented by a charge k on L. Then, according to [**T9**, Chap. X, §5.4], we have:

$$q_{M,\sigma}([m_j]) = \frac{1}{2} - \frac{k_j}{2b_{jj}} \bmod 1.$$

Substituting $k_j = 1 - s_j + \sum_i b_{ij}$, we find that this formula agrees with (4.2) of Example 4.4. This prove the theorem in this particular case. Now consider the general case, when L is not necessary algebraically split. We shall use the following lemma (due to T. Ohtsuki).

LEMMA 4.19. [**De2**] *Let M be a rational homology 3-sphere. There exist then non-zero integers $n_1, \ldots, n_r$ such that $M\#L(n_1,1)\#\cdots\#L(n_r,1)$ can be presented by surgery along a framed link L algebraically split in $\mathbf{S}^3$.*

Here $\#$ denotes connected sum and $L(n,1)$ is the 3-dimensional lens space obtained by surgery along a trivial knot with framing $n \in \mathbb{Z}^\times$ in $\mathbf{S}^3$. Set $M' = M\#L(n_1,1)\#\cdots\#L(n_r,1)$ and set $\sigma' = \sigma\#\sigma_1\#\cdots\#\sigma_r \in \mathrm{Spin}^c(M')$ where $\sigma_1, \ldots, \sigma_r$ denote arbitrary Spin^c-structures on the lens spaces. Then, $q_{M',\sigma'} = \phi_{M',\sigma'}$. There is an orthogonal splitting $\phi_{M',\sigma'} = \phi_{M,\sigma} \oplus \phi_{L(n_1,1),\sigma_1} \oplus \cdots \oplus \phi_{L(n_r,1),\sigma_r}$ (as follows easily from the surgery formula for these quadratic functions). The same holds for $q_{M,\sigma}$ (as follows from the behaviour of the maximal Abelian Reidemeister-Turaev torsion under $\#$: see [**T9**, Chap. XII, §1]). Then, for any $x \in H_1(M)$, we have: $q_{M,\sigma}(x) = q_{M',\sigma'}(i_*(x)) = \phi_{M',\sigma'}(i_*(x)) = \phi_{M,\sigma}(x)$ and we are done. □

5. Goussarov-Habiro theory for 3-manifolds with complex spin structures

5.1. A brief review of the Y-equivalence relation. The Goussarov-Habiro theory for compact oriented 3-manifolds is a finite type invariants theory based on an elementary move called Y-surgery.

DEFINITION 5.1. Let M be a compact oriented 3-manifold. A *Y-graph* G in M is an (unoriented) embedding of the surface drawn in Figure 5.1, together with its decomposition between *leaves*, *edges* and *node*.
Let $j : H_3 \hookrightarrow M$ be a positive embedding of the genus 3 handlebody onto a regular

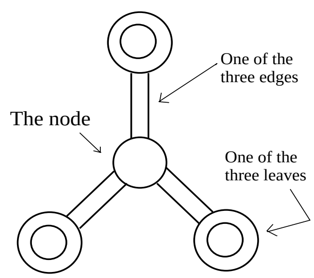

Figure 5.1: A Y-graph.

neighborhood N(G) of G in M. The compact oriented 3-manifold *obtained from M by Y-surgery along G* is defined as:

$$M_G = M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial} (H_3)_L\,,$$

where $(H_3)_L$ is the surgered handlebody along the six-component link[8] L drawn on Figure 5.2.
We call *Y-equivalence* the equivalence relation among compact oriented 3-manifolds generated by Y-surgeries and positive diffeomorphisms.

REMARK 5.2. A Y-surgery, introduced by Goussarov in [**Go**], is equivalent to a *A_1-move*, defined by Habiro in [**Hr**], or to a *Borromean surgery*, introduced by Matveev in [**Mt**].

[8] We use the blackboard framing convention.

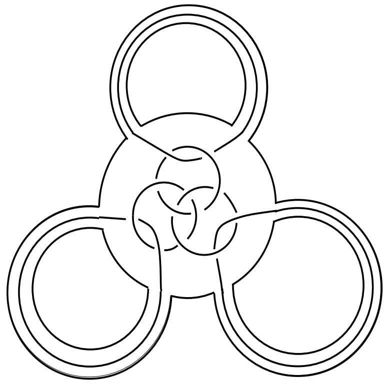

Figure 5.2: The surgery meaning of a Y-graph.

Observe that M_G comes with an inclusion $M \setminus \mathrm{int}\,(\mathrm{N}(G)) \hookrightarrow M_G$ and that, due to the choice of $\mathrm{N}(G)$ and its trivialization j, M_G is only defined up to some particular positive diffeomorphisms.
As it is well-known, a Y-surgery is a twist in the following sense:

Remark 5.3. There is a diffeomorphism:

$$M_G \cong M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial \circ h} H_3, \tag{5.1}$$

which is the identity on $M \setminus \mathrm{int}\,(\mathrm{N}(G))$. Here, $h : \Sigma_3 \to \Sigma_3$, called the *Borromean* diffeomorphism, is a specific self-diffeomorphism of $\Sigma_3 = \partial H_3$ which acts trivially in homology (details can be found in §2 of Chapt. 2).

5.2. The Y^c-equivalence relation.

5.2.1. *Twist and* Spin^c*-structures.* Let us consider again the situation which was under study in §3.3. We thus work with two compact oriented 3-manifolds M_1 and M_2 and a positive diffeomorphism $f : -\partial M_2 \to \partial M_1$, and we form the closed oriented 3-manifold:

$$M = M_1 \;\cup_f\; M_2.$$

Here, we make the additional assumption that $\partial M_2 \cong \partial M_1$ *is connected.*
Let also $h : \partial M_2 \to \partial M_2$ be a positive diffeomorphism *which acts trivially in homology*, and the manifold M' be obtained from M by twisting it along ∂M_2 with h:

$$M' = M_1 \;\cup_{f \circ h}\; M_2. \tag{5.2}$$

By a Mayer-Vietoris argument, there is a canonical homology isomorphism $\Phi : H_1(M) \to H_1(M')$ which is unambiguously defined by the following commutative diagram:

$$\begin{tikzcd}
 & H_1(M) \arrow[dd, "\simeq"', "\Phi"] & \\
H_1(M_1) \arrow[ur, "j_{1,*}"] \arrow[dr, "j'_{1,*}"'] & & H_1(M_2) \arrow[ul, "j_{2,*}"'] \arrow[dl, "j'_{2,*}"] \\
 & H_1(M') &
\end{tikzcd}$$

where j_1, j_2, j'_1 and j'_2 denote inclusions.

PROPOSITION 5.4. *The above described twist $M \rightsquigarrow M'$ induces a canonical bijection:*

$$\mathrm{Spin}^c(M) \xrightarrow[\simeq]{\Omega} \mathrm{Spin}^c(M')$$

which is affine over $P\Phi P^{-1}: H^2(M) \to H^2(M')$. Moreover, the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Spin}^c(M) \arrow[r, "\Omega"] \arrow[d, "c"'] & \mathrm{Spin}^c(M') \arrow[d, "c"] \\ H^2(M) \arrow[r, "P\Phi P^{-1}"'] & H^2(M'). \end{tikzcd}$$

PROOF. Take $\alpha \in \mathrm{Spin}^c(M)$, $\Omega(\alpha)$ is then defined as follows. *Pick* a Spin-stucture σ_2 on ∂M_2 and let $\sigma_1 = f_*(-\sigma_2) \in \mathrm{Spin}(\partial M_1)$. Since h_* at the level of $H_1(\partial M_2; \mathbb{Z}_2)$ is the identity, h acts trivially on $\mathrm{Spin}(\partial M_2)$: this follows from the fact that Johnson's correspondance $\mathrm{Spin}(\partial M_2) \to \mathrm{Quad}\,(\bullet)$ is natural (§3.2.5). According to Corollary 3.29, there are then two gluing maps:

$$\mathrm{Spin}^c(M_1, \sigma_1) \times \mathrm{Spin}^c(M_2, \sigma_2) \xrightarrow{\cup_f} \mathrm{Spin}^c(M),$$

$$\mathrm{Spin}^c(M_1, \sigma_1) \times \mathrm{Spin}^c(M_2, \sigma_2) \xrightarrow{\cup_{f\circ h}} \mathrm{Spin}^c(M'),$$

which are respectively affine, *via* the Poincaré dualities, over $j_{1,*} \oplus j_{2,*}$ and $j'_{1,*} \oplus j'_{2,*}$. Since ∂M_2 is connected, the map $\cup_f$ is then surjective. *Pick* $\alpha_1 \in \mathrm{Spin}^c(M_1, \sigma_1)$ and $\alpha_2 \in \mathrm{Spin}^c(M_2, \sigma_2)$ such that $\alpha = \alpha_1 \cup_f \alpha_2$, form:

$$\alpha' = \alpha_1 \cup_{fh} \alpha_2 \in \mathrm{Spin}^c(M').$$

and define $\Omega(\alpha)$ to be α'.

Let us now verify that $\Omega(\alpha)$ is well-defined by that procedure. Assume other intermediate choices $\tilde{\sigma}_2$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ of respectively σ_2, α_1 and α_2, leading to $\tilde{\alpha}' = \tilde{\alpha}_1 \cup_{fh} \tilde{\alpha}_2$. We claim that $\alpha' = \tilde{\alpha}'$.

Assume first the particular case when: $\tilde{\sigma}_2 = \sigma_2 \in \mathrm{Spin}(\partial M_2)$. Then, since $\alpha_1 \cup_f \alpha_2 = \alpha = \tilde{\alpha}_1 \cup_f \tilde{\alpha}_2$, we have

$$j_{1,*} P^{-1}\left(\frac{\alpha_1}{\tilde{\alpha}_1}\right) + j_{2,*} P^{-1}\left(\frac{\alpha_2}{\tilde{\alpha}_2}\right) = P^{-1}\left(\frac{\alpha}{\alpha}\right) = 0 \in H_1(M).$$

Applying Φ to that identity, we obtain this equation:

$$j'_{1,*} P^{-1}\left(\frac{\alpha_1}{\tilde{\alpha}_1}\right) + j'_{2,*} P^{-1}\left(\frac{\alpha_2}{\tilde{\alpha}_2}\right) = 0 \in H_1(M'),$$

whose lhs equals $P^{-1}(\alpha'/\tilde{\alpha}')$ because $\alpha' = \alpha_1 \cup_{fh} \alpha_2$ and $\tilde{\alpha}' = \tilde{\alpha}_1 \cup_{fh} \tilde{\alpha}_2$. We then conclude that $\alpha' = \tilde{\alpha}'$.

We now deal with the general case. Let $s_2 \in \mathrm{Spin}_r(\partial M_2)$ represent σ_2 and let $s_1 = f_*(-s_2) \in \mathrm{Spin}_r(\partial M_1)$, which then represents σ_1. For $i = 1$ and 2, take $a_i \in \mathrm{Spin}^c_r(M_i)$ representing α_i and such that $a_i|_{\partial M_i} = \beta(s_i)$. We then have:

$$\alpha = [a_1 \cup_f a_2] \in \mathrm{Spin}^c(M), \tag{5.3}$$

where $\cup_f$ denotes here the rigid version of the gluing map (that of Lemma 3.28). Pick a homotopy between $h^*(s_2)$ and s_2, and let $U \in \mathrm{Spin}_r(\partial M_2 \times I)$ be the corresponding rigid Spin-struture on the product. Then,

$$\alpha' = [a_1 \cup_{fh} (\beta(U) \cup a_2)] \in \mathrm{Spin}^c(M'). \tag{5.4}$$

Let also $\tilde{s}_2 \in \mathrm{Spin}_r(\partial M_2)$ represent $\tilde{\sigma}_2$ and let $\tilde{s}_1 = f_*(-\tilde{s}_2) \in \mathrm{Spin}_r(\partial M_1)$ which then represent $\tilde{\sigma}_1 := f_*(-\tilde{\sigma}_2) \in \mathrm{Spin}(\partial M_1)$. Even if σ_2 and $\tilde{\sigma}_2$ may be different, we certainly have: $\beta(\sigma_2) = \beta(\tilde{\sigma}_2) \in \mathrm{Spin}^c(\partial M_2)$. Choose an homotopy between $\beta(\tilde{s}_2)$ and $\beta(s_2)$, and let $H \in \mathrm{Spin}^c_r(\partial M_2 \times I)$ be the corresponding rigid structure on the product. Then, $f_*(-H) \in \mathrm{Spin}^c_r(\partial M_1 \times I)$ is a homotopy between $\beta(s_1)$ and $\beta(\tilde{s}_1)$. Now, the rigid Spin^c-structure on M:

$$(a_1 \cup f_*(-H)) \cup_f (H \cup a_2)$$

can be homotoped, by means of the double collar of $j_2(\partial M_2)$ in M, to $a_1 \cup_f a_2$. The gluing map:

$$\mathrm{Spin}^c(M_1, \tilde{\sigma}_1) \times \mathrm{Spin}^c(M_2, \tilde{\sigma}_2) \xrightarrow{\cup_f} \mathrm{Spin}^c(M),$$

then sends according to (5.3), the pair $([a_1 \cup f_*(-H)], [H \cup a_2])$ to α. According to the particular case treated previously, and whatever the choices of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ have been, we have:

$$\tilde{\alpha}' = [a_1 \cup f_*(-H)] \cup_{fh} [H \cup a_2].$$

Let $V \in \mathrm{Spin}_r(\partial M_2 \times I)$ be a homotopy between $h^*(\tilde{s}_2)$ and $\tilde{s}_2$. We obtain:

$$\tilde{\alpha}' = [(a_1 \cup f_*(-H)) \cup_{fh} (\beta(V) \cup H \cup a_2)]. \tag{5.5}$$

Just as before for M, the rigid Spin^c-structure on M':

$$(a_1 \cup f_*(-H)) \cup_{fh} (h^*(H) \cup \beta(U) \cup a_2)$$

can be homotoped, by means of the double collar of $j_2'(\partial M_2)$ in M', to $a_1 \cup_{fh} (\beta(U) \cup a_2)$. According to (5.4), we then have:

$$\alpha' = [(a_1 \cup f_*(-H)) \cup_{fh} (h^*(H) \cup \beta(U) \cup a_2)]. \tag{5.6}$$

By (5.5) and (5.6), the identity $\tilde{\alpha}' = \alpha'$ will follow from this equality of Spin^c-structures relative to Spin-structures:

$$[\beta(V) \cup H] = [h^*(H) \cup \beta(U)] \in \mathrm{Spin}^c(\partial M_2 \times I, \tilde{\sigma}_2 \dot{\cup} (-\sigma_2)). \tag{5.7}$$

Since $H^2(\partial M_2 \times I, \partial(\partial M_2 \times I))$ has no 2-torsion, these relative Spin^c-structures are determined by their Chern classes. We have:

$$c([\beta(U)]) = 0 \quad \text{and:} \quad c([\beta(V)]) = 0,$$

by Remark 3.25, and $c([h^*(H)]) = h^*(c([H]))$ (by naturality of obstructions) and so $c([h^*(H)]) = c([H])$. By the last statement of Corollary 3.29, (5.7) is then satisfied. We thus conclude that the map Ω is well-defined.

Let us now verify that Ω is affine. Take $\alpha \in \mathrm{Spin}^c(M)$ and $x \in H_1(M)$. Pick a Spin-structure σ_2 on ∂M_2 define $\sigma_1 = f_*(-\sigma_2)$ and write α as:

$$\alpha = \alpha_1 \cup_f \alpha_2 \in \mathrm{Spin}^c(M),$$

where $\alpha_i \in \mathrm{Spin}^c(M_i, \sigma_i)$. Write also x as $x = j_{1,*}(x_1) + j_{2,*}(x_2)$ where $x_i \in H_1(M_i)$. Then, $P(x) \cdot \alpha = (P(x_1) \cdot \alpha_1) \cup_f (P(x_2) \cdot \alpha_2)$, hence:

$$\begin{aligned}
\Omega(P(x) \cdot \alpha) &= (P(x_1) \cdot \alpha_1) \cup_{fh} (P(x_2) \cdot \alpha_2) \\
&= P\left(j'_{1,*}(x_1) + j'_{2,*}(x_2)\right) \cdot (\alpha_1 \cup_{fh} \alpha_2) \\
&= P\left(\Phi j_{1,*}(x_1) + \Phi j_{2,*}(x_2)\right) \cdot \Omega(\alpha) \\
&= P\Phi(x) \cdot \Omega(\alpha).
\end{aligned}$$

Finally, using the same notations:

$$\begin{aligned}
P^{-1}c(\Omega(\alpha)) &= P^{-1}c(\alpha_1 \cup_{fh} \alpha_2) \\
&= j'_{1,*}\left(P^{-1}c(\alpha_1)\right) + j'_{2,*}\left(P^{-1}c(\alpha_2)\right) \\
&= \Phi\left(j_{1,*}\left(P^{-1}c(\alpha_1)\right) + j_{2,*}\left(P^{-1}c(\alpha_2)\right)\right) \\
&= \Phi\left(P^{-1}(c(\alpha))\right) \in H_1(M').
\end{aligned}$$

□

REMARK 5.5. When M has non-empty boundary (*i.e.* when M_1 and M_2 have been glued along only a connected part of their boundaries), there is a relative version of Prop. 5.4 (involving Spin^c-structures on M and M' relative to Spin-structures).

5.2.2. *Definition of the Y^c-surgery.* We now explain how a Y-surgery makes sense in the setting of Spin^c-manifolds.

LEMMA 5.6. *Let M be a closed oriented 3-manifold and let G be a Y-graph in M. The Y-surgery $M \rightsquigarrow M_G$ then induces a canonical bijection:*

$$\mathrm{Spin}^c(M) \xrightarrow[\simeq]{\Omega_G} \mathrm{Spin}^c(M_G)$$

denoted by $\alpha \mapsto \alpha_G$. The map Ω_G is affine over $P\Phi_G P^{-1}$, where $\Phi_G : H_1(M) \to H_1(M_G)$ is the homology isomorphism which is unambiguously defined by the following commutative diagram:

$$\begin{tikzcd}
 & H_1(M) \arrow[dd, "\simeq"', "\Phi_G"] \\
H_1(M \setminus \mathrm{N}(G)) \arrow[ur, "k_*"] \arrow[dr, "k'_*"'] & \\
 & H_1(M_G).
\end{tikzcd}$$

Here, $\mathrm{N}(G)$ *is a regular neighborhood of G in M, $k : M \setminus \mathrm{N}(G) \hookrightarrow M$ and $k' : M \setminus \mathrm{N}(G) \hookrightarrow M_G$ denote the inclusions.*

DEFINITION 5.7. The Spin^c-manifold (M_G, α_G) is said to be obtained from the Spin^c-manifold (M, α) by *Y^c-surgery* along the graph G.
We call *Y^c-equivalence* the equivalence relation among closed Spin^c-manifolds of dimension 3 generated by Y^c-surgeries and Spin^c-diffeomorphisms.

PROOF OF LEMMA 5.6. We fix a positive embedding $j : H_3 \hookrightarrow M$ onto a regular neighborhood $\mathrm{N}(G)$ of G in M. By Remark 5.3, there exists a self-diffeomorphism h of $\Sigma_3 = \partial H_3$ which has the property to act trivially in homology and to be such that there exists a diffeomorphism:

$$M_G = M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial} (H_3)_L \xrightarrow[\simeq]{f} M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial \circ h} H_3,$$

which is the identity on $M \setminus \mathrm{int}\,(\mathrm{N}(G))$. There is then a bijection:

$$\mathrm{Spin}^c\,(M_G) \xrightarrow[\simeq]{f_*} \mathrm{Spin}^c\left(M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial \circ h} H_3\right),$$

Also, by §5.2.1, there is a canonical bijection:

$$\mathrm{Spin}^c\,(M) \xrightarrow[\simeq]{\Omega} \mathrm{Spin}^c\left(M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial \circ h} H_3\right).$$

We then define Ω_G to be the composite $f_*^{-1}\Omega$. We now have to verify that the bijection Ω_G does not depend on the intermediate choices.

First, we prove the independance on the pair (h, f) with the above property. Let (h', f') be another choice, it suffices to prove that the following diagram is commutative:

$$\begin{tikzcd}
& \mathrm{Spin}^c\left(M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial \circ h} H_3\right) \arrow[dd, "g_*"] \\
\mathrm{Spin}^c(M) \arrow[ur, "\Omega"] \arrow[dr, "\Omega'"'] & \\
& \mathrm{Spin}^c\left(M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial \circ h'} H_3\right),
\end{tikzcd}$$

where $g = f' \circ f^{-1}$. Let $\alpha \in \mathrm{Spin}^c(M)$. Pick a $\sigma \in \mathrm{Spin}(\Sigma_3)$, and write α as $\alpha_1 \cup j_*(\alpha_2)$ where $\alpha_1 \in \mathrm{Spin}^c\left(M \setminus \mathrm{int}\,(\mathrm{N}(G)), -j_*(\sigma)\right)$ and $\alpha_2 \in \mathrm{Spin}^c(H_3, \sigma)$. Then, $\Omega(\alpha) = \alpha_1 \cup_{j|_\partial \circ h} \alpha_2$ and $\Omega'(\alpha) = \alpha_1 \cup_{j|_\partial \circ h'} \alpha_2$. We have: $g_*\left(\Omega(\alpha)\right) = \alpha_1 \cup_{j|_\partial \circ h'} \left(g|\right)_* (\alpha_2)$. Here, $\left(g|\right)_* : \mathrm{Spin}^c(H_3, \sigma) \to \mathrm{Spin}^c(H_3, \sigma)$ is the identity since $g|_{H_3}$ acts trivially in homology and elements of $\mathrm{Spin}^c(H_3, \sigma)$ are classified by their relative Chern classes. It follows that: $g_*\left(\Omega(\alpha)\right) = \Omega'(\alpha)$.

Second, we prove the independance on j. Let j' be a trivialization of another regular neighborhood $\mathrm{N}'(G)$ of G in M, and let $(q_t : M \to M)_{t \in I}$ be an ambiant isotopy between j and j': $q_0 = \mathrm{Id}_M$ and $q_1 \circ j = j'$. Let $q : M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial \circ h} H_3 \to M \setminus \mathrm{int}\left(\mathrm{N}'(G)\right) \cup_{j'|_\partial \circ h} H_3$ be the positive diffeomorphism induced by q_1. Proving "the independance on j" means proving that the following diagram commutes:

$$\begin{tikzcd}
& \mathrm{Spin}^c\left(M \setminus \mathrm{int}\,(\mathrm{N}(G)) \cup_{j|_\partial \circ h} H_3\right) \arrow[dd, "q_*"] \\
\mathrm{Spin}^c(M) \arrow[ur, "\Omega"] \arrow[dr, "\Omega'"'] & \\
& \mathrm{Spin}^c\left(M \setminus \mathrm{int}\left(\mathrm{N}'(G)\right) \cup_{j'|_\partial \circ h} H_3\right).
\end{tikzcd}$$

Let $\alpha \in \mathrm{Spin}^c(M)$ and, as above, write it as $\alpha = \alpha_1 \cup j_*(\alpha_2)$. Since q_1 is isotopic to the identity, it acts trivially on $\mathrm{Spin}^c(M)$, then $\alpha = (q_1)_*(\alpha) = \left(q_1|\right)_*(\alpha_1) \cup j'_*(\alpha_2)$. We then have: $\Omega(\alpha) = \alpha_1 \cup_{j|_\partial \circ h} \alpha_2$ and $\Omega'(\alpha) = \left(q_1|\right)_*(\alpha_1) \cup_{j'|_\partial \circ h} \alpha_2$, so that: $q_*\left(\Omega(\alpha)\right) = \Omega'(\alpha)$. ☐

By Remark 5.5, we could also define a Y^c-surgery move between compact oriented 3-manifolds with non-empty boundaries, provided they are equipped with Spin^c-structures relative to Spin-structures.

The Y^c-surgery is then the elementary move of a Spin^c-refinement of the Goussarov-Habiro finite type invariants theory. Furthermore, it can be shown that the calculus of *clovers* developped in [**GGP**] extends to the context of Spin^c-manifolds.

REMARK 5.8. It has been introduced in Chap. 2 a Spin-refinement of the Goussarov-Habiro finite type invariants theory. In particular, it is shown that a Y-surgery induces a canonical bijection $\Theta_G : \mathrm{Spin}(M) \to \mathrm{Spin}\,(M_G)$. Both refinements are compatible, in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Spin}(M) \arrow[r, "\Theta_G", "\simeq"'] \arrow[d, "\beta"'] & \mathrm{Spin}\,(M_G) \arrow[d, "\beta"] \\
\mathrm{Spin}^c(M) \arrow[r, "\simeq", "\Omega_G"'] & \mathrm{Spin}^c(M_G).
\end{tikzcd}$$

5.2.3. *A combinatorial formulation of the Y^c-equivalence relation.* Given a surgery equivalence relation among closed oriented 3-manifolds, it can sometimes be derived from unknotting operations and surgery presentations in $\mathbf{S}^3$. By §3.4.2, the same could be done for closed Spin^c-manifolds of dimension 3. We now formulate the Y^c-equivalence relation in this way, taking the Δ-*move* of [**MN**] as unknotting operation.

LEMMA 5.9. *The Y^c-equivalence relation is the equivalence relation among closed* Spin^c*-manifolds of dimension* 3 *generated by* Spin^c*-diffeomorphisms and Δ^c-moves. Here, a Δ^c-move is defined to be the move depicted on Figure 5.3 between surgery presentations of closed* Spin^c*-manifolds of dimension* 3.

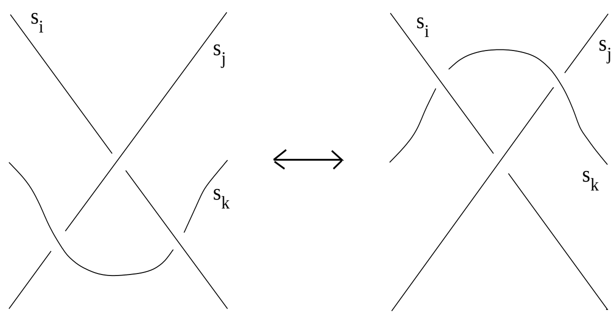

Figure 5.3: A Δ^c-move.

PROOF. Let us begin with a closed connected oriented 3-manifold M and a Y-graph G in M. Suppose also that $M \cong V_L$ is presented by surgery along a n-component ordered oriented framed link L in $\mathbf{S}^3$. Isotope G in M so that it becomes disjoint from the dual link to L, then $G \subset \mathbf{S}^3 \setminus L$. Take a regular neighborhood of G in $\mathbf{S}^3 \setminus L$, and put into this genus 3 handlebody the 2-component framed link K depicted on Figure 5.4. The link K can be obtained from the link of Figure 5.2 by

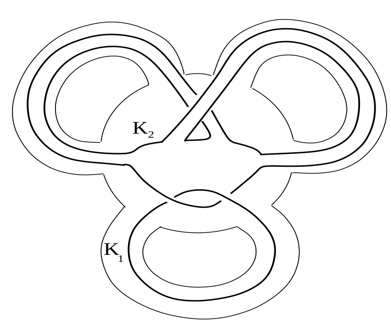

Figure 5.4: Y-surgery as surgery along a 2-component link.

some slam dunk moves (see Example 3.34) and some handle-slide moves in H_3. In particular, $V_{L\cup K}$ is a surgery presentation of M_G. Using the viewpoint from §3.4.2, we now want to give the combinatorial analog of the bijection Ω_G. In other words, we want to recognize the map O_G defined by the following commutative diagram:

$$\begin{tikzcd} \mathcal{S}^c_L \arrow[r, dotted, "O_G"] & \mathcal{S}^c_{L\cup K} \\ \mathrm{Spin}^c(M) \arrow[u, "\simeq"] \arrow[r, "\Omega_G"'] & \mathrm{Spin}^c\,(M_G). \arrow[u, "\simeq"'] \end{tikzcd}$$

If B_L denotes the linking matrix of L and if K is appropriately oriented, then the ordered union of ordered oriented framed links $L \cup K$ has a linking matrix shaped as:

$$B_{L\cup K} = \left(\begin{array}{ccc|cc} & & & x_1 & 0 \\ & B_L & & \vdots & \vdots \\ & & & x_l & 0 \\ \hline x_1 & \cdots & x_l & x & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right).$$

CLAIM 5.10. We have:

$$\forall [s] \in \mathcal{S}^c_L, \quad O_G\,([s]) = [(s, x, 0)] \in \mathcal{S}^c_{L\cup K}. \tag{5.8}$$

This will be the key point to prove the lemma.

PROOF OF CLAIM 5.10. As pointed out in Remark 5.8, a Y-surgery along G also induces a bijection $\Omega_G : \mathrm{Spin}(M) \to \mathrm{Spin}(M_G)$, a combinatorial analog of which is also given in Chap. 2. Using the compatibility stated in Remark 5.8 and §3.4.3, we see that equation (5.8) holds at least for those elements of $\mathcal{S}^c_L$ which come from $\mathcal{S}_L$.

Denote by (H, f) the lattice corresponding to the intersection pairing on W_L, and by (H', f') that of $W_{L\cup K}$. Recall from Remark 3.35 that there are canonical isomorphisms $H^2\,(V_L) \simeq \mathrm{Coker}\ \widehat{f}$ and $H^2\,(V_{L\cup K}) \simeq \mathrm{Coker}\ \widehat{f'}$. The isomorphism $P\Phi_G P^{-1} : H^2(M) \to H^2\,(M_G)$ corresponds then to the isomorphism $\mathrm{Coker}\ \widehat{f} \to \mathrm{Coker}\ \widehat{f'}$ defined by $[y] \mapsto [(y, 0, 0)]$.

Take now $[s] \in \mathcal{S}^c_L$ arising from $\mathcal{S}_L$ and let $[y] \in \mathbb{Z}^n/\mathrm{Im}\ B_L \simeq \mathrm{Coker}\widehat{f}$. We aim to calculate $O_G\,([y]\cdot[s]) \in \mathcal{S}^c_{L\cup K}$. The $\cdot$ here corresponds to the combinatorial description of the action of $H^2\,(V_L)$ on $\mathrm{Spin}^c(V_L)$ given in Remark 3.35. The map Ω_G being affine over $P\Phi_G P^{-1}$, we have: $O_G\,([y]\cdot[s]) = [(y, 0, 0)] \cdot O_G\,([s])$ $= [(y, 0, 0)] \cdot [(s, x, 0)] = [(s + 2y, x, 0)]$. Therefore, equation (5.8) also holds for $[y] \cdot [s]= [s + 2y]$. The transitivity of the action of $H^2(V_L)$ on $\mathrm{Spin}^c(V_L)$ allows us to conclude. □

On the one hand, a Δ^c-move can be realized by a Y^c-surgery as shown in Figure 5.5. In this sequence of Spin^c-diffeomorphisms, the first is obtained by by applying Claim 5.10, while the second is obtained from a handle-slide move and a slam dunk move (Example 3.34).

On the other hand, a Y^c-surgery can be realized by a Δ^c-move, as shown in Figure 5.6. In this depicted sequence, the first Spin^c-diffeomorphism is obtained from three slam dunk moves. Then, a Δ^c-move is applied. Next Spin^c-diffeomorphism is obtained by Spin^c- calculi (in particular, two slam dunks have been performed), and the last one is obtained from Claim 5.10. □

5.3. Proof of Theorem 3. Let (M, σ) and (M', σ') be two closed Spin^c-manifolds of dimension 3.

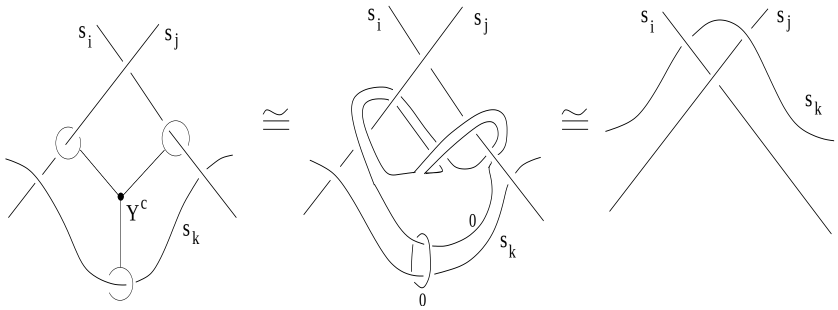

Figure 5.5: A Δ^c-move can be realized by a Y^c-surgery.

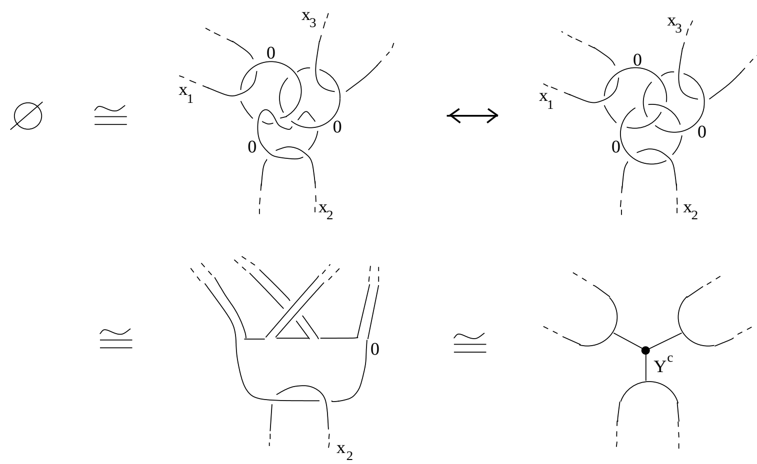

Figure 5.6: A Y^c-surgery can be realized by a Δ^c-move

5.3.1. *Proof of the equivalence* (2) $\Longleftrightarrow$ (3).

Lemma 5.11. *Let $\psi : H_1(M) \to H_1(M')$ be an isomorphism, and let $\psi^\sharp : H_2(M';\mathbb{Q}/\mathbb{Z}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$ be the isomorphism dual to ψ with respect to the intersection pairings. The following assertions are then equivalent:*

(a) $L_{M'} = L_M \circ \left(\psi^\sharp\right)^{\otimes 2}$;
(b) $\lambda_M = \lambda_{M'} \circ (\psi|)^{\otimes 2}$;
(c) *the following diagram is commutative:*

$$\begin{array}{ccc} H_2\left(M';\mathbb{Q}/\mathbb{Z}\right) & \xrightarrow{B} & \mathrm{T}H_1(M') \\ \psi^\sharp \downarrow \simeq & & \psi| \uparrow \simeq \\ H_2\left(M;\mathbb{Q}/\mathbb{Z}\right) & \xrightarrow{B} & \mathrm{T}H_1(M). \end{array}$$

Proof. For any $x_1', x_2' \in H_2(M';\mathbb{Q}/\mathbb{Z})$, we have:

$$L_{M'}\left(x_1', x_2'\right) = \lambda_{M'}\left(B(x_1'), B(x_2')\right) = B(x_1') \bullet x_2', \text{ and:}$$

$$L_M\left(\psi^\sharp(x_1'), \psi^\sharp(x_2')\right) = B\psi^\sharp(x_1') \bullet \psi^\sharp(x_2') = \psi B \psi^\sharp(x_1') \bullet x_2'.$$

Then, the equivalence between (a) and (c) follows from the left nondegeneracy of the intersection pairing[9] $\bullet : H_1(M') \times H_2(M';\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ in M'.
For any $y_1 \in \mathrm{T}H_1(M)$ and any $x_2 \in H_2(M;\mathbb{Q}/\mathbb{Z})$, we have:

$$\lambda_M\left(y_1, B(x_2)\right) = y_1 \bullet x_2 = \psi(y_1) \bullet \left(\psi^\sharp\right)^{-1}(x_2) = \lambda_{M'}\left(\psi(y_1), B\left(\psi^\sharp\right)^{-1}(x_2)\right),$$

to be compared with $\lambda_{M'}\left(\psi(y_1), \psi B(x_2)\right)$. Thus, the nondegeneracy of λ'_M implies the equivalence between (b) and (c). $\square$

Suppose that the condition (2) of Th. 3 is satisfied. Then, $L_{M'} = L_M \circ \left(\psi^\sharp\right)^{\otimes 2}$ and so $\lambda_M = \lambda_{M'} \circ (\psi|)^{\otimes 2}$ by Lemma 5.11. Moreover, since we have the relation $d_{\phi_{M',\sigma'}} = d_{\phi_{M,\sigma}} \circ \psi^\sharp$ between homogeneity defects of quadratic functions, we obtain by Lemma 4.9: $\langle c(\sigma'), x'\rangle = \langle c(\sigma), \psi^\sharp(x')\rangle$, and so: $P^{-1}c(\sigma') \bullet x' = \psi P^{-1}c(\sigma) \bullet x'$ for any $x' \in H_2(M';\mathbb{Q}/\mathbb{Z})$. By left nondegeneracy of $\bullet : H_1(M') \times H_2(M';\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$, we obtain: $P^{-1}c(\sigma') = \psi P^{-1}c(\sigma)$. Finally, the quadratic function:

$$\phi_{M,\sigma} s = \phi_{M,\sigma} \psi^\sharp s' \psi| = \phi_{M',\sigma'} s' \psi$$

is isomorphic to $\phi_{M',\sigma'} s'$: hence, these quadractic functions have identical Gauss sums. Condition (3) then holds.

Conversely, suppose that condition (3) of Th. 3 is satisfied. First, the short exact sequence:

$$0 \longrightarrow H_2(M) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H_2(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{B} \mathrm{T}H_1(M) \longrightarrow 0$$

is split, $H_2(M) \otimes \mathbb{Q}/\mathbb{Z} = \mathrm{Ker}\ \widehat{L_M}$ and $\mathrm{T}H_1(M)$ is a finite Abelian group: thus, the pair $\left(H_2(M;\mathbb{Q}/\mathbb{Z}), \phi_{M,\sigma}\right)$ meets the finiteness condition of §2.4.
From Lemma 5.11, we deduce that $L_{M'} = L_M \circ \left(\psi^\sharp\right)^{\otimes 2}$. Also, from Lemma 4.7 and from Lemma 4.9, we respectively deduce that $r_{\phi_{M',\sigma'}} = r_{\phi_{M,\sigma}} \circ \psi^\sharp$ and $d_{\phi_{M',\sigma'}} = d_{\phi_{M,\sigma}} \circ \psi^\sharp$. Since $\psi| \circ B \circ \psi^\sharp = B$ (by Lemma 5.11), the sections s and s' are $\psi^\sharp$-compatible in the sense of Definition 2.20. By applying Corollary 2.22, we thus obtain that the quadratic functions $\Phi_{M,\sigma}$ and $\Phi_{M',\sigma'}$ are isomorphic. More precisely, there exists an isomorphism[10] $\varphi : \mathrm{T}H_1(M) \to \mathrm{T}H_1(M')$ such that $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \varphi_1$, where $\varphi_1 : H_2(M';\mathbb{Q}/\mathbb{Z}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$ corresponds to the isomorphism:

$$\left(\psi^\sharp \otimes \mathrm{Id}_{\mathbb{Q}/\mathbb{Z}}\right) \times \varphi^{-1} : \left(H_2(M') \otimes \mathbb{Q}/\mathbb{Z}\right) \times \mathrm{T}H_1(M') \to \left(H_2(M) \otimes \mathbb{Q}/\mathbb{Z}\right) \times \mathrm{T}H_1(M)$$

via the identifications:

$$\begin{array}{rcl} H_2(M';\mathbb{Q}/\mathbb{Z}) & \simeq & \left(H_2(M') \otimes \mathbb{Q}/\mathbb{Z}\right) \times \mathrm{T}H_1(M') \\ H_2(M;\mathbb{Q}/\mathbb{Z}) & \simeq & \left(H_2(M) \otimes \mathbb{Q}/\mathbb{Z}\right) \times \mathrm{T}H_1(M) \end{array}$$

respectively defined by the sections s' and s. By Lemma 2.13 (together with the 4-dimensional definition of $\phi_{M,\sigma}$ and $\phi_{M',\sigma'}$ from §4.1.2), there exists an isomorphism[11] $\eta : H_1(M) \to H_1(M')$ such that $\eta^\sharp = \varphi_1$. Consequently: $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \eta^\sharp$.

[9] The left nondegeneracy of $\bullet$ is the injectivity of the map $\mu_M : H^2(M) \to \mathrm{Hom}\left(H_2\left(M;\mathbb{Q}/\mathbb{Z}\right), \mathbb{Q}/\mathbb{Z}\right)$ introduced by Th. 4.10.

[10] φ may be different from $\psi|$.

[11] Observe that η does not necessarily coincides with ψ, because $\eta| : \mathrm{T}H_1(M) \to \mathrm{T}H_1(M')$ is equal to φ. Nevertheless, we certainly have: $\psi^\sharp = \eta^\sharp : H_2(M') \to H_2(M)$.

5.3.2. *Proof of the equivalence* (1) $\Longleftrightarrow$ (2). We start by proving implication (1) $\Longrightarrow$ (2). By Lemma 5.9, it suffices to prove it when (M,σ) and (M',σ') are related by a Spin^c-diffeomorphism or, for some fixed surgery presentations of them, by a Δ^c-move.
The first case is obvious. Indeed, let $f:(M,\sigma)\to(M',\sigma')$ be a Spin^c-diffeomorphism. We then easily deduce that $\phi_{M,\sigma}=\phi_{M',\sigma'}\circ f_*$ from the intrinsic definition of these quadratic functions (§4.4); note that $(f_*)^{-1}=(f_*)^\sharp$.
The second case is deduced from the 4-dimensional definition of the quadratic functions $\phi_{M,\sigma}$ and $\phi_{M',\sigma'}$ (§4.1.2), and from the fact that a Δ-move between ordered oriented links preserve their linking matrices.

Suppose now that condition (2) is satisfied. We fix some surgery presentations $V_L\cong M$ and $V_{L'}\cong M'$. As in §3.4, we put $H=H_2(W_L)$ (resp. $H'=H_2(W_{L'})$) and $f:H\times H\to\mathbb{Z}$ (resp. $f':H'\times H'\to\mathbb{Z}$) will denote the intersection pairing in W_L (resp. $W_{L'}$). Let also $c\in\mathrm{Char}(f)$ and $c'\in\mathrm{Char}(f')$ represent respectively σ and σ'. Then, by hypothesis, the quadratic functions $\phi_{f,c}:G_f\to\mathbb{Q}/\mathbb{Z}$ and $\phi_{f',c'}:G_{f'}\to\mathbb{Q}/\mathbb{Z}$ are isomorphic *via* an isomorphism which is induced by an isomorphism Coker $\widehat{f}\to$ Coker $\widehat{f'}$. Theorem 2.11 then says that the bilinear lattices with characteristic forms (H,f,c) and (H',f',c') are stably equivalent. Also, by Corollary 2.14, we can restrict ourselves to stabilizations with copies of $(\mathbb{Z},\pm1,1)$. An isomorphism of bilinear lattices (resp. a stabilization by $(\mathbb{Z},\pm1,1)$) can topologically be realized by a finite sequence of Spin^c Kirby's moves: handle slidings and changes of orientation of components (resp. by a stabilization by a ±1-framed unknot). We can then suppose, without loss of generality, that $(H,f,c)=(H',f',c')$. Concretely, that means that the linking matrices B_L and $B_{L'}$ are equal and that there is a multi-integer s such that $[s]\in\mathcal{S}^c_L$ represent σ and $[s]\in\mathcal{S}^c_{L'}$ represent σ'.
A theorem of Murakami and Nakanishi [**MN**, Theorem 1.1] (or Matveev [**Mt**]), states that two ordered oriented links have identical linking matrices if, and only if, they are Δ-equivalent. Then, the "decorated links" (L,s) and (L',s) are Δ-equivalent: therefore, by Lemma 5.9, the Spin^c-manifolds (M,σ) and (M',σ') are Y^c-equivalent.

REMARK 5.12. Observe that the present proof allows for a more precise statement of the equivalence (1) $\Longleftrightarrow$ (2) of Th. 3.
Indeed, any path of Spin^c-diffeomorphisms and Y^c-surgeries:

$$(M,\sigma)=(M_0,\sigma_0)\rightsquigarrow(M_1,\sigma_1)\rightsquigarrow(M_2,\sigma_2)\rightsquigarrow\cdots\rightsquigarrow(M_n,\sigma_n)=(M',\sigma')$$

yields an isomorphism $\psi:H_1(M)\to H_1(M')$. It is the composite of the isomorphisms $H_1(M_i)\to H_1(M_{i+1})$, which is taken to be f_* if the step $(M_i,\sigma_i)\rightsquigarrow(M_{i+1},\sigma_{i+1})$ is a Spin^c-diffeomorphism f, or is the isomorphism Φ_G of Lemma 5.6 if this step is the Y^c-surgery along a Y-graph $G\subset M_i$. Then, ψ verifies: $\phi_{M',\sigma'}=\phi_{M,\sigma}\circ\psi^\sharp$. Conversely, given an isomorphism $\psi:H_1(M)\to H_1(M')$ with the property that $\phi_{M',\sigma'}=\phi_{M,\sigma}\circ\psi^\sharp$, we can find a path from (M,σ) to (M',σ') inducing ψ at the level of $H_1(-)$.

CHAPITRE 5

Torsions abéliennes de Reidemeister-Turaev

1. Rappels sur les torsions abéliennes de Reidemeister-Turaev

Nous présentons dans cette section les torsions abéliennes de Reidemeister telles qu'elles ont été raffinées par Turaev. L'objectif premier est de parvenir à la définition des torsions abéliennes des $Spin^c$-variétés fermées de dimension 3.

Nos références seront [**T7**], [**T4**], [**T8**] et [**T3**]. Le lecteur pourra trouver une présentation générale de la théorie des torsions combinatoires dans [**T7**], et notamment les références historiques et bibliographiques les concernant.

Conventions 1.1. Si G est un groupe abélien, un *G-espace affine* (ou *espace affine au-dessus de* G) sera un ensemble sur lequel G agit librement et transitivement. Pour ces actions, la notation multiplicative sera adoptée.

Conventions 1.2. Les CW-complexes sont supposés connexes et finis et, par défaut, leurs *cellules* sont des cellules ouvertes.

1.1. Torsion d'un complexe de chaînes. Dans cette section, nous rappelons les bases algébriques de la théorie des torsions abéliennes. $\mathbf{F}$ désignera un corps commutatif.

1.1.1. *Définitions basiques.* Soit V un $\mathbf{F}$-espace vectoriel et soient b et c des bases de V. On note alors :

$$[b/c] \in \mathbf{F}^* = \mathbf{F} \setminus \{0\}$$

le déterminant de la représentation matricielle de b dans la base c. Les bases b et c sont dite *équivalentes*, et on note $b \sim c$, lorsque $[b/c] = 1$; la relation $\sim$ est une relation d'équivalence.

Soient une suite exacte courte de $\mathbf{F}$-espaces vectoriels :

$$0 \longrightarrow V' \xrightarrow{\ i\ } V \xrightarrow{\ j\ } V'' \longrightarrow 0,$$

ainsi que c' et c'' des bases respectives de V' et V''. On note alors $c'c''$ la classe d'équivalence de bases de V obtenue par concaténation de $i(c')$ avec un relèvement par j de c''. La classe $c'c''$ ne dépend que de celles de c' et de c''.

Soit maintenant :

$$C = (C_m \xrightarrow{\partial_{m-1}} C_{m-1} \longrightarrow \cdots \xrightarrow{\partial_0} C_0)$$

un $\mathbf{F}$-complexe de chaînes de dimension finie et de longueur m. On suppose que C est *basé* et *homologiquement basé*, ce qui signifie respectivement que :

- pour tout $i \in \{0, \dots, m\}$, C_i vient avec une base préférée c_i, on note alors $c = (c_0, \dots, c_m)$;
- pour tout $i \in \{0, \dots, m\}$, $H_i(C)$ vient avec une base préférée h_i, on note alors $h = (h_0, \dots, h_m)$.

Nous définissons les entiers modulo 2 suivants :

$$\alpha_i(C) = \sum_{k=0}^{i} \dim(C_k) \in \mathbf{Z}_2, \tag{1.1}$$

$$\beta_i(C) = \sum_{k=0}^{i} \dim(H_k(C)) \in \mathbf{Z}_2, \tag{1.2}$$

$$N(C) = \sum_{k=0}^{m} \alpha_k(C) \cdot \beta_k(C) \in \mathbf{Z}_2. \tag{1.3}$$

Choisissons pour tout $i \in \{0, \dots, m\}$ une base b_i de $B_i(C)$. Les suites exactes de $\mathbf{F}$-espaces vectoriels :

$$0 \longrightarrow B_i(C) \longrightarrow Z_i(C) \longrightarrow H_i(C) \longrightarrow 0,$$

$$\text{et :} \quad 0 \longrightarrow Z_i(C) \longrightarrow C_i \xrightarrow{\partial_{i-1}} B_{i-1}(C) \longrightarrow 0,$$

permettent, suivant le procédé décrit plus haut, de construire une classe d'équivalence

$$b_i h_i b_{i-1} := (b_i h_i) b_{i-1}$$

de bases du $\mathbf{F}$-espace vectoriel C_i.

Definition 1.3. (Turaev, [**T7**, §1 et §18]). La *torsion* du $\mathbf{F}$-complexe C, basé par c et homologiquement basé par h, est :

$$\tau(C; c, h) = \prod_{i=0}^{m} [b_i h_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbf{F}^*.$$

Sa *torsion raffinée* est :

$$\check{\tau}(C; c, h) = (-1)^{N(C)} \cdot \tau(C; c, h) \in \mathbf{F}^*.$$

On vérifie que ces quantités ne dépendent pas du choix des b_i, et ne dépendent donc que de C, c et h. Remarquons aussi que lorsque C est acyclique, on a : $\check{\tau}(C; c, \varnothing) = \tau(C; c, \varnothing)$.

1.1.2. *Multiplicativité de la torsion.* Soit une suite exacte courte :

$$O \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

de $\mathbf{F}$-complexes de longueur m et de dimension finie. On suppose les complexes C', C et C'' homologiquement basés par respectivement h', h et h'' et basés par respectivement c', c et c''. On suppose c', c et c'' *compatibles* dans le sens où :

$$\forall i \in \{0, \dots, m\},\ c_i \sim c'_i c''_i.$$

En particulier, la suite longue d'homologie associée à cette s.e.c. de $\mathbf{F}$-complexes :

$$\mathcal{H} := \left(H_m(C') \longrightarrow H_m(C) \longrightarrow H_m(C'') \longrightarrow \cdots \longrightarrow H_0(C) \longrightarrow H_0(C'')\right),$$

est, comme $\mathbf{F}$-complexe, basée par $(h'_m, h_m, h''_m, \dots, h_0, h''_0)$.

Théorème 1.4. (Turaev, [**T3**, Lemma 3.4.2]) *Sous les hypothèses et notations précédentes, nous avons alors :*

$$\check{\tau}(C; c, h) = (-1)^{\theta+\nu} \tau\left(\mathcal{H}; (h'_m, h_m, h''_m, \dots), \varnothing\right) \cdot \check{\tau}(C'; c', h') \cdot \check{\tau}(C''; c'', h'') \tag{1.4}$$

où θ et ν sont les entiers modulo 2 définis par :

$$\theta = \sum_{i=0}^{m} \left((\beta_i(C) + 1) \left(\beta_i(C') + \beta_i(C'')\right) + \beta_{i-1}(C')\, \beta_i(C'') \right) \in \mathbf{Z}_2,$$

$$\nu = \sum_{i=0}^{m} \alpha_i(C'')\, \alpha_{i-1}(C') \in \mathbf{Z}_2.$$

Spécialisons maintenant cette formule de multiplicativité à deux cas particuliers qui nous intéresseront.

EXEMPLE 1.5. Plaçons-nous dans le cas où C'' est acyclique, et supposons les bases homologiques h et h' *compatibles*, en ce sens où elles se correspondent par l'isomorphisme $H_*(C') \xrightarrow{\simeq} H_*(C)$. On a alors en particulier : $\beta_i(C'') = 0, \beta_i(C') = \beta_i(C)$, d'où $\theta = 0$. On vérifie aussi facilement que $\tau\left(\mathcal{H}; (h'_m, h_m, \varnothing, \dots), \varnothing\right) = 1$. La formule (1.4) s'écrit alors plus simplement :

$$\check{\tau}(C; c, h) = (-1)^\nu \cdot \check{\tau}(C'; c', h') \cdot \tau(C''; c'', \varnothing) \in \mathbf{F}^*. \tag{1.5}$$

EXEMPLE 1.6. Plaçons-nous dans le cas où $C = C' \oplus C''$ et où, pour tout i, $c_i = (c'_i, c''_i)$ est la base de C obtenue par concaténation de la famille ordonnée $c'_i \subset C' \subset C$ avec la famille ordonnée $c''_i \subset C'' \subset C$ (de sorte que les bases c', c et c'' sont compatibles au sens précédent). On a en particulier : $H_*(C) = H_*(C') \oplus H_*(C'')$, et on suppose que $h_i = (h'_i, h''_i)$ est, pour tout i, la concaténation de h'_i avec h''_i. La suite longue d'homologie $\mathcal{H}$ a tous ses connectants nuls et on vérifie que :

$$\tau\left(\mathcal{H}; (h'_m, h_m, h''_m, \dots), \varnothing\right) = 1.$$

De plus, $\beta_i(C) = \beta_i(C') + \beta_i(C'')$ d'où :

$$\theta = \sum_{i=0}^{m} \beta_{i-1}\left(C'\right) \beta_i\left(C''\right) \in \mathbf{Z}_2.$$

On peut donc retenir la formule :

$$\check{\tau}\left(C' \oplus C''; (c', c''), (h', h'')\right) = (-1)^\rho \cdot \check{\tau}\left(C'; c', h'\right) \cdot \check{\tau}\left(C''; c'', h''\right), \tag{1.6}$$

où ρ est l'entier modulo 2 défini par[1] :

$$\rho = \sum_{i=0}^{m} \left(\alpha_i\left(C''\right)\alpha_{i-1}\left(C'\right) + \beta_i\left(C''\right)\beta_{i-1}\left(C'\right)\right) \in \mathbf{Z}_2.$$

1.2. Orientations homologiques et structures d'Euler combinatoires. Les raffinements de Turaev des torsions abéliennes de Reidemeister nécessitent de fixer des structures supplémentaires sur les espaces. Ce sont les *orientations homologiques* et les *structures d'Euler combinatoires.*

1.2.1. *Orientations homologiques.* Soit X un espace topologique compact.

DEFINITION 1.7. (Turaev, [**T7**, §18]). Une *orientation homologique* ω de X est une orientation du $\mathbf{R}$-espace vectoriel :

$$H_*(X; \mathbf{R}) = \bigoplus_{i \geq 0} H_i(X; \mathbf{R})$$

L'orientation homologique opposée à ω sera notée $-\omega$.

EXEMPLE 1.8. Soit M une 3-variété fermée orientée et connexe. Son orientation détermine une orientation homologique :

$$\omega_M = \left([pt], b, b^\sharp, [M]\right)$$

où $[pt]$ est la classe d'homologie d'un point de M, $[M]$ est sa classe fondamentale, b est une base de $H_1(M; \mathbf{R})$ et $b^\sharp$ est la base de $H_2(M; \mathbf{R})$ duale de b pour la forme d'intersection de M. L'orientation ω_M est appelée l'*orientation homologique canonique* de la variété fermée orientée connexe M.

[1]Ce nombre ne fait pas jouer des rôles symétriques à C' et C'', ce qui s'accorde avec le fait que le signe de $\check{\tau}\left(C' \oplus C''; (c', c''), (h', h'')\right)$ n'est en général pas égal à celui de $\check{\tau}\left(C'' \oplus C'; (c'', c'), (h'', h')\right)$.

1.2.2. *Structures d'Euler combinatoires sur un CW-complexe.* Soit K un CW-complexe[2] de support $X = |K|$. On suppose que $\chi(X) = 0$.

DEFINITION 1.9. (Turaev, [**T4**, §1]) Une *chaîne d'Euler* sur K est une 1-chaîne singulière ξ de X dont le bord est de la forme :

$$\partial\xi = \sum_{\alpha,\ \text{cellule de } K} (-1)^{\dim(\alpha)} c_\alpha$$

où c_α est un point de la cellule α.

L'hypothèse faite sur $\chi(X)$ assure l'existence des chaînes d'Euler. Soient maintenant ξ et η des chaînes d'Euler sur K :

$$\partial\xi = \sum_{\alpha,\ \text{cellule de } K} (-1)^{\dim(\alpha)} c_\alpha \quad \text{et :} \quad \partial\eta = \sum_{\alpha,\ \text{cellule de } K} (-1)^{\dim(\alpha)} d_\alpha.$$

Choisissons un chemin γ_α, contenu dans α et joignant c_α à d_α, et formons le 1-cycle :

$$\xi - \eta + \sum_{\alpha,\ \text{cellule de } K} (-1)^{\dim(\alpha)} \gamma_\alpha.$$

La classe élément de $H_1(X)$ qu'il représente ne dépend pas des choix des γ_α. Les chaînes d'Euler ξ et η sont dites *équivalentes*, et on note $\xi \sim \eta$, lorsque cette classe d'homologie s'annule. La relation $\sim$ est une relation d'équivalence.

DEFINITION 1.10. (Turaev, [**T4**, §1]) Une *structure d'Euler combinatoire* sur K est une classe d'équivalence de chaînes d'Euler. On note $Eul(K)$ l'espace des structures d'Euler combinatoires sur K.

Par l'addition des 1-chaînes, $H_1(X)$ agit sur $Eul(K)$: l'action est libre et transitive. Donc, $Eul(K)$ est un $H_1(X)$-espace affine.

LEMME 1.11. (Turaev, [**T4**, §1.2]) *Soit K' une subdivision de K. Il existe alors une bijection canonique et $H_1(X)$-équivariante :*

$$Eul(K) \xrightarrow[\simeq]{\sigma(K,K')} Eul(K').$$

De plus, si K'' est une subdivision de K', nous avons alors :

$$\sigma(K, K'') = \sigma(K', K'') \circ \sigma(K, K').$$

1.2.3. *Structures d'Euler combinatoires sur une* 3*-variété lisse fermée orientée.* Le Lemme 1.11 permet de définir successivement :

– d'abord, l'espace des structures d'Euler combinatoires :

$$Eul(P)$$

sur un polyèdre P de caractéristique d'Euler $\chi(P) = 0$, par identification avec $Eul(K)$ pour une triangulation (K, ρ) de P ; on note :

$$Eul(K) \xrightarrow[\simeq]{\rho_*} Eul(P),$$

cette identification (voir [**T4**, §2.1]) ;

– puis, l'espace des structures d'Euler combinatoires :

$$Eul(M)$$

sur une variété M lisse et de caractéristique d'Euler $\chi(M) = 0$, par identification avec $Eul(K)$ pour une C^1-triangulation (K, ρ) de M ; on note :

$$Eul(K) \xrightarrow[\simeq]{\rho_*} Eul(M),$$

cette identification (voir [**T4**, §2.2]).

[2]On rappelle que dans ce chapitre, les CW-complexes sont supposés finis et connexes.

Nous nous restreignons ici aux variétés de dimension 3 lisses fermées et orientées.

Théorème 1.12. (Turaev, [**T7**, Th. 20.2] ou [**T4**, §6]) *Soit M une 3-variété lisse fermée et orientée. Il existe alors une bijection canonique :*

$$Eul(M) \xrightarrow{\simeq} vect(M),$$

entre ses structures d'Euler combinatoires et ses structures d'Euler géométriques définies au §2.2.3 du Chapitre 1. De plus, cette identification est affine au-dessus de l'isomorphisme $H_1(M) \simeq H^2(M)$ donné par la dualité de Poincaré.

On déduit alors du Lemme 2.20 du Chapitre 1, qu'il existe une bijection canonique et affine :

$$Eul(M) \xrightarrow{\simeq} Spin^c(M)$$

entre les structures d'Euler combinatoires de M et ses $Spin^c$-structures.

1.2.4. *Revêtement abélien maximal d'un espace pointé.* Afin d'introduire dans le prochain paragraphe la notion de *famille fondamentale de cellules*, nous fixons quelques conventions concernant les revêtements abéliens maximaux.

Soit $(X,\star)$ un espace topologique pointé qui est supposé séparé, connexe par arcs, localement connexe par arcs et semi-localement simplement connexe. Nous notons $\left(\tilde{X},q\right)$ son revêtement universel défini par

$$\tilde{X} = \left\{\text{classes d'homotopie rel } \partial I \text{ de chemins } I=[0,1] \xrightarrow{\gamma} X \text{ tels que } \gamma(0)=\star\right\}$$

$$\text{et } q\left([\gamma]\right) = \gamma(1),$$

une topologie adéquate étant donnée à $\tilde{X}$. Notons que $\tilde{X}$ peut être basé par $\tilde{\star}$, le chemin identiquement égal à $\star$. Le point $\tilde{\star}$ défini alors un isomorphisme canonique $\pi_1(X,\star) \xrightarrow{\simeq} Aut\left(\tilde{X},q\right)$. Soit $\left(\widehat{X},p\right)$ le revêtement obtenu en quotientant $\left(\tilde{X},q\right)$ par le sous-groupe de $Aut\left(\tilde{X},q\right)$ correspondant au sous-groupe dérivé $\pi_1(X,\star)'$. Le groupe $Aut\left(\widehat{X},p\right)$ s'identifie alors canoniquement à

$$H_1(X) \simeq \pi_1(X,\star)/\pi_1(X,\star)'.$$

Definition 1.13. Le revêtement $\left(\widehat{X},p\right)$ est appelé le *revêtement abélien maximal* de l'espace pointé $(X,\star)$.

Dans la suite, nous adopterons les conventions suivantes :

- L'espace $\widehat{X}$ a un point base préféré : noté $\widehat{\star}$, c'est la classe du point $\tilde{\star}$ précedemment défini.
- Si $(Y,\star)$ est un autre espace topologique pointé (supposé séparé, connexe par arcs, localement connexe par arcs et semi-localement simplement connexe) et si $(X,\star) \xrightarrow{f} (Y,\star)$ est une application continue préservant les points bases, nous noterons $\widehat{X} \xrightarrow{\widehat{f}} \widehat{Y}$ l'unique relèvement de f défini par $f\left(\widehat{\star}\right) = \widehat{\star}$.

1.2.5. *Familles fondamentales de cellules.* Dans ce paragraphe, nous considérons un CW-complexe pointé

$$(K,\star)$$

de support $|K| = X$ et de caractéristique d'Euler $\chi(X)$ quelconque. Soit $\left(\widehat{X},p\right)$ le revêtement abélien maximal de $(X,\star)$. La décomposition cellulaire K de X se relève en une décomposition cellulaire $\widehat{K}$ de $\widehat{X}$.

DEFINITION 1.14. (Turaev, [**T7**, §20.1]) Une famille $\widehat{e}$ de cellules de $\widehat{K}$ est dite *fondamentale* lorsqu'elle contient exactement un relèvement de chaque cellule de K. Dans ce cas, si σ est une cellule de K, on note $\widehat{e}(\sigma)$ l'unique élément de $\widehat{e}$ dont la projection par p est σ.

Etant données des familles fondamentales de cellules $\widehat{e}$ et $\widehat{f}$ de $\widehat{K}$, on peut former :

$$\frac{\widehat{f}}{\widehat{e}} = \prod_{\sigma,\ \text{cellule de } K} \left(\frac{\widehat{f}(\sigma)}{\widehat{e}(\sigma)}\right)^{(-1)^{\dim(\sigma)}} \in H_1(X),$$

où $\widehat{f}(\sigma)/\widehat{e}(\sigma)$ désigne l'unique élément h de $H_1(X) \simeq Aut\left(\widehat{X}, p\right)$ tel que $h \cdot \widehat{e}(\sigma) = \widehat{f}(\sigma)$. Les familles fondamentales $\widehat{e}$ et $\widehat{f}$ sont dites *équivalentes*, et on note $\widehat{e} \sim \widehat{f}$, lorsque $\widehat{f}/\widehat{e} = 1 \in H_1(X)$. La relation $\sim$ est une relation d'équivalence. L'espace-quotient est noté :

$$E(K, \star).$$

Soit $\mathfrak{e} \in E(K,\star)$ une classe représentée par la famille fondamentale de cellules $\widehat{e}$, et soit $h \in H_1(X)$; on note alors $h \cdot \mathfrak{e}$ la classe d'équivalence d'une famille fondamentale $\widehat{f}$ vérifiant $\widehat{f}/\widehat{e} = h \in H_1(X)$. Il découle des définitions qu'on définit de la sorte une action de $H_1(X)$ sur $E(K,\star)$ et, que par cette action, $E(K,\star)$ est un $H_1(X)$-espace affine.

LEMME 1.15. (Turaev, [**T7**, Lemma 20.1]) *Supposons que K' est une subdivision cellulaire de K. Il existe alors une bijection canonique et $H_1(X)$-équivariante :*

$$E(K,\star) \xrightarrow[\simeq]{\sigma(K,K')} E(K',\star).$$

De plus, si K'' est une subdivision de K', nous avons alors :

$$\sigma(K,K'') = \sigma(K',K'') \circ \sigma(K,K').$$

DÉMONSTRATION. Soit $\widehat{e}$ une famille fondamentale de cellules de $\widehat{K}$; on note alors $\widehat{e}'$ la famille fondamentale de cellules de $\widehat{K'}$ constituées des cellules qui sont contenues dans une cellule élément de $\widehat{e}$. Si $\widehat{f}$ est une autre famille fondamentale de cellules de $\widehat{K}$ donnant lieu à $\widehat{f}'$ suivant ce procédé-là, alors :

$$\frac{\widehat{f}'}{\widehat{e}'} = \frac{\widehat{f}}{\widehat{e}} \in H_1(X),$$

comme il découle du fait que si σ est une cellule de K de fermeture $\overline{\sigma}$, alors :

$$\begin{aligned} \sum_{\substack{\alpha \in K' \\ \alpha \subset \sigma}} (-1)^{\dim(\alpha)} &= \chi(\overline{\sigma}) - \chi(\partial\sigma) \\ &= 1 - \left(1 + (-1)^{\dim(\sigma)-1}\right) \\ &= (-1)^{\dim(\sigma)}. \end{aligned}$$

On en déduit que l'assignation $[\widehat{e} \longmapsto \widehat{e}']$ définit bien une application $\sigma(K,K')$ de $E(K,\star)$ vers $E(K',\star)$ et qui est, pour la même raison, $H_1(X)$-équivariante. □

LEMME 1.16. (Turaev, [**T4**, §1.3]) *Supposons que $\chi(X) = 0$. Il existe alors une bijection canonique et $H_1(X)$-équivariante :*

$$E(K,\star) \xrightarrow[\simeq]{} Eul(K)$$

De plus, si K' est une subdivision de K, le diagramme suivant commute :

$$\begin{array}{ccc} E(K,\star) & \xrightarrow[\simeq]{} & Eul(K) \\ {\scriptstyle \sigma(K,K')}\downarrow & & \downarrow{\scriptstyle \sigma(K,K')} \\ E(K',\star) & \xrightarrow[\simeq]{} & Eul(K'). \end{array}$$

REMARQUE 1.17. Le Lemme 1.16 montre en particulier que, lorsque $\chi(X) = 0$, l'espace $E(K,\star)$ ne dépend essentiellement pas du choix du point base $\star$.

PREUVE DU LEMME 1.16. Soit $\mathfrak{e} \in E(K,\star)$ représentée par la famille fondamentale de cellules $\widehat{e}$. Considérons un point de $\widehat{X}$: par exemple le point base préféré $\widehat{\ast}$ défini au §1.2.4. Pour toute cellule $\sigma \in K$ on choisit un chemin allant de $\widehat{\star}$ à un point de $\widehat{e}(\sigma)$ si $\dim(\sigma)$ est paire, ou d'un point de $\widehat{e}(\sigma)$ à $\widehat{\star}$ si $\dim(\sigma)$ est impaire. La réunion de ces chemins est une 1-chaîne singulière de $\widehat{X}$ qui se projette par p en une chaîne d'Euler ξ sur K. La bijection annoncée entre $E(K,\star)$ et $Eul(K)$ est induite par cette assignation $[\widehat{e} \longmapsto \xi]$. □

1.3. Torsion d'un CW-complexe. Nous sommes maintenant en mesure de définir les torsions abéliennes de Reidemeister des CW-complexes, telles qu'elles ont été raffinées par Turaev.

1.3.1. *En caractéristique d'Euler quelconque.* Soit K un CW-complexe de support $|K| = X$ et de caractéristique d'Euler quelconque. Donnons-nous aussi $\mathbf{F}$ un corps commutatif, et un homomorphisme d'anneaux :

$$\mathbf{Z}[H_1(X)] \xrightarrow{\varphi} \mathbf{F}.$$

On suppose fixés :

(1) une orientation homologique ω de X,

(2) un point base $\star \in X$,

(3) un élément $\mathfrak{e}$ de $E(K,\star)$,

(4) une famille $b = (b_0, b_1, \dots)$ où b_i est une base de $H_i^{\varphi}(X,\star)$.

Ici, $H_i^{\varphi}(X,\star)$ désigne le $i^{\text{ème}}$ *groupe d'homologie à coefficients φ-twistés de l'espace pointé* $(X,\star)$. Il est défini par

$$H_i^{\varphi}(X,\star) := H_i\left(C_*^{\varphi}(K,\star)\right),$$

où $C_*^{\varphi}(K,\star)$ est le $\mathbf{F}$-complexe $\mathbf{F} \otimes_{\mathbf{Z}[H_1(X)]} C_*\left(\widehat{K}\right)$, $\left(\widehat{X},p\right)$ désignant le revêtement abélien maximal de $(X,\star)$ et $\mathbf{F}$ étant muni de la structure de $\mathbf{Z}[H_1(X)]$-module donnée par φ. Notons que si $\star$ et $\star'$ sont deux points distincts de X, les groupes $H_i^{\varphi}(X,\star)$ et $H_i^{\varphi}(X,\star')$ sont alors (non-canoniquement) isomorphes.

Nous allons associer à la liste précédente une torsion qui sera notée :

$$\breve{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega) \in \mathbf{F}^*.$$

Nous préciserons ensuite la dépendance de cette quantité en chacun des items de cette liste-là. Voici la recette définissant cette quantité :

(1) on *o*riente et on *o*rdonne les cellules de K d'une façon arbitraire : ce choix sera noté *oo* ;

(2) on choisit une famille fondamentale $\widehat{e}$ de cellules de $\widehat{K}$ représentant $\mathfrak{e}$;

(3) les choix de oo et de $\widehat{e}$ déterminent une $\mathbf{Z}[H_1(X)]$-base de $C_*\left(\widehat{K}\right)$, puis une $\mathbf{F}$-base de $C_*^{\varphi}(K,\star)$, qu'on note $\widehat{e}_{oo}$; on calcule la torsion raffinée :

$$\check{\tau}\left(C_*^{\varphi}(K,\star)\,;\widehat{e}_{oo},b\right);$$

(4) le choix oo détermine une $\mathbf{R}$-base du complexe $C_*(K;\mathbf{R})$; on choisit aussi une base $w=(w_0,w_1,\dots)$ de $H_*(X;\mathbf{R})$ compatible avec l'orientation homologique ω de X ; on calcule la torsion raffinée :

$$\check{\tau}\left(C_*(K;\mathbf{R});oo,w\right);$$

(5) on pose finalement :

$$\check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega):=\operatorname{sgn}\left(\check{\tau}\left(C_*(K;\mathbf{R});oo,w\right)\right)\cdot\check{\tau}\left(C_*^{\varphi}(K,\star)\,;\widehat{e}_{oo},b\right)\in\mathbf{F}^*.$$

Lemme 1.18. (Turaev) *La quantité* $\check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega)$ *possède les propriétés suivantes.*
a) Elle ne dépend pas des choix intermédiaires faits.
b) Elle satisfait aux relations :

$$\begin{aligned}\check{\tau}^{\varphi}(K,\star;h\cdot\mathfrak{e},b,\omega)&=\varphi(h)\cdot\check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega)\,,\\ \check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,-\omega)&=-\check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega)\,.\end{aligned}$$

c) Elle est invariante par subdvision ; précisément, si K' est une subdivision de K, nous avons :

$$\check{\tau}^{\varphi}(K',\star;\sigma(K,K')(\mathfrak{e}),b,\omega)=\check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega)\,.$$

Démonstration. Prouvons les assertions a) et b). D'abord, lorsqu'on permute deux cellules de K ou lorsqu'on inverse l'orientation d'une cellule de K lors du choix oo, les deux quantités $\operatorname{sgn}\left(\check{\tau}\left(C_*(K;\mathbf{R});oo,w\right)\right)$ et $\check{\tau}\left(C_*^{\varphi}(K,\star)\,;\widehat{e}_{oo},b\right)$ changent simultanément de signe, ce qui prouve l'indépendance de $\check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega)$ en le choix oo.
Soit ensuite $\widehat{f}$ une autre famille fondamentale de cellules de $\widehat{K}$. Alors, il découle des définitions que :

$$\check{\tau}\left(C_*^{\varphi}(K,\star)\,;\widehat{f}_{oo},b\right)=\varphi\left(\frac{\widehat{f}}{\widehat{e}}\right)\cdot\check{\tau}\left(C_*^{\varphi}(K,\star)\,;\widehat{e}_{oo},b\right).$$

Ceci prouve l'indépendance de $\check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega)$ en le choix de $\widehat{e}$ dans la classe $\mathfrak{e}$ ainsi que la première relation de b).
Soit enfin $v=(v_0,v_1,\dots)$ un autre choix de base de $H_*(X;\mathbf{R})$. Il découle aussi des définitions que :

$$\operatorname{sgn}\left(\check{\tau}\left(C_*(K;\mathbf{R});oo,v\right)\right)=\left(\prod_{i\geq 0}\operatorname{sgn}[v_i/w_i]\right)\cdot\operatorname{sgn}\left(\check{\tau}\left(C_*(K;\mathbf{R});oo,w\right)\right).$$

Cette égalité prouve l'indépendance de $\check{\tau}^{\varphi}(K,\star;\mathfrak{e},b,\omega)$ en le choix de w dans la classe ω ainsi que la seconde relation de l'assertion b).
L'assertion c) est prouvée dans [**T4**, Lemma 3.2.3][3]. □

[3] L'énoncé de ce lemme utilise les structures d'Euler combinatoires plutôt que les familles fondamentales de cellules, et suppose donc que $\chi(X)=0$. La preuve de ce lemme ne nécessite en fait pas cette hypothèse.

1.3.2. *En caractéristique d'Euler nulle.* Soit K un CW-complexe de support $|K| = X$ et de caractéristique d'Euler $\chi(X) = 0$. Soient aussi $\mathbf{F}$ un corps commutatif et $\mathbf{Z}[H_1(X)] \xrightarrow{\varphi} \mathbf{F}$ un homomorphisme d'anneaux.

LEMME 1.19. *Soient $\star, \star' \in X$ et supposons la condition d'acyclicité*

$$H_*^\varphi(X, \star) = 0 = H_*^\varphi(X, \star')$$

satisfaite. Soient ω une orientation homologique de X, $e \in Eul(K)$ et $\mathfrak{e} \in E(K, \star)$ (resp. $\mathfrak{e}' \in E(K, \star')$) correspondant à la structure d'Euler combinatoire e par le Lemme 1.16. Nous avons alors :

$$\check{\tau}^\varphi(K, \star; \mathfrak{e}, \varnothing, \omega) = \check{\tau}^\varphi(K, \star'; \mathfrak{e}', \varnothing, \omega).$$

DÉMONSTRATION. Soit $\left(\widehat{X}, p\right)$ et $\left(\widehat{X}', p'\right)$ les revêtements abéliens maximaux respectifs de $(X, \star)$ et $(X, \star')$. Piochons une transformation de revêtements

$$\left(\widehat{X}, p\right) \xrightarrow{\phi} \left(\widehat{X}', p'\right),$$

induisant une flèche $E(K, \star) \xrightarrow{\phi} E(K, \star')$ de la façon évidente. Il découle alors de la définition des identifications $E(K, \star) \simeq Eul(K)$ et $E(K, \star') \simeq Eul(K)$ (voir la preuve du Lemme 1.16) que $\phi(\mathfrak{e}) = \mathfrak{e}'$. La transformation ϕ induit alors un isomorphisme de $\mathbf{F}$-complexes :

$$\mathbf{F} \otimes_{\mathbf{Z}[H_1(X)]} C_*\left(\widehat{K}\right) \underset{\simeq}{\longrightarrow} \mathbf{F} \otimes_{\mathbf{Z}[H_1(X)]} C_*\left(\widehat{K}'\right)$$

transportant la base $\widehat{e}_{oo}$ sur la base $\widehat{e}'_{oo}$, où *oo* est un choix d'*o*rdre et d'*o*rientations des cellules de K, $\widehat{e}$ est un représentant de $\mathfrak{e}$ et $\widehat{e}' = \phi(\widehat{e})$ (qui représente donc $\mathfrak{e}'$). D'où :

$$\check{\tau}\left(C_*^\varphi(K, \star); \widehat{e}_{oo}, \varnothing\right) = \check{\tau}\left(C_*^\varphi(K, \star'); \widehat{e}'_{oo}, \varnothing\right).$$

□

Ce lemme justifie la définition suivante.

DEFINITION 1.20. (Turaev, [**T7**, §20.1], [**T4**, §3.2] ou [**T8**, §4.2]) La *torsion* associée à φ *du CW-complexe* K, muni de l'orientation homologique ω et de la structure d'Euler combinatoire e, est :

$$\tau^\varphi(K; e, \omega) = \begin{cases} \check{\tau}^\varphi(K, \star; \mathfrak{e}, \varnothing, \omega) \in \mathbf{F}^* & \text{si } H_*^\varphi(X, \star) = 0, \\ 0 \in \mathbf{F} & \text{si } H_*^\varphi(X, \star) \neq 0, \end{cases}$$

où $\star \in X$ et où $\mathfrak{e} \in E(K, \star)$ correspond à la structure d'Euler combinatoire e.

Il découle alors directement du Lemme 1.18 que $\tau^\varphi(K; e, \omega)$ est invariante par subdivision, soit :

$$\tau^\varphi(K; e, \omega) = \tau^\varphi\left(K'; \sigma(K, K')(e), \omega\right) \in \mathbf{F},$$

qu'elle vérifie :

$$\tau^\varphi(K; h \cdot e, \omega) = \varphi(h) \cdot \tau^\varphi(K; e, \omega) \in \mathbf{F}$$

ainsi que :

$$\tau^\varphi(K; e, -\omega) = -\tau^\varphi(K; e, \omega) \in \mathbf{F}.$$

1.4. Torsion d'une *Spin*^{c}-variété fermée de dimension 3. Il découle des lignes concluant le précédent paragraphe et de §1.2.3 que les deux définitions qui suivent sont légitimes.

DEFINITION 1.21. (Turaev, [**T4**, §3]) Soient P un polyèdre de caractéristique d'Euler nulle, $\mathbf{F}$ un corps commutatif et $\mathbf{Z}[H_1(P)] \xrightarrow{\varphi} \mathbf{F}$ un homomorphisme d'anneaux.
Alors, la *torsion* associée à φ *du polyèdre* P, muni de l'orientation homologique ω et de la structure d'Euler combinatoire e, est :

$$\tau^{\varphi}(P; e, \omega) := \tau^{\varphi\circ\rho_*}\left(K; \rho_*^{-1}(e), \rho_*^{-1}(\omega)\right) \in \mathbf{F},$$

où (K, ρ) est une triangulation du polyèdre P.

DEFINITION 1.22. (Turaev, [**T4**, §3]) Soient M une variété lisse de caractéristique d'Euler nulle, $\mathbf{F}$ un corps commutatif et $\mathbf{Z}[H_1(M)] \xrightarrow{\varphi} \mathbf{F}$ un homomorphisme d'anneaux.
Alors, la *torsion* associée à φ *de la variété lisse* M, munie de l'orientation homologique ω et de la structure d'Euler combinatoire e, est :

$$\tau^{\varphi}(M; e, \omega) := \tau^{\varphi\circ\rho_*}\left(K; \rho_*^{-1}(e), \rho_*^{-1}(\omega)\right) \in \mathbf{F},$$

où (K, ρ) est une C^1-triangulation de la variété M.

Cette dernière définition se spécialise à la suivante.

DEFINITION 1.23. (Turaev, [**T5**]) Soient (M, σ) une *Spin*^{c}-variété fermée de dimension 3, $\mathbf{F}$ un corps commutatif et $\mathbf{Z}[H_1(M)] \xrightarrow{\varphi} \mathbf{F}$ un homomorphisme d'anneaux.
Alors, la *torsion* associée à φ *de la Spin*^{c}*-variété fermée* (M, σ) est :

$$\tau^{\varphi}(M, \sigma) := \tau^{\varphi}(M; e, \omega_M) \in \mathbf{F},$$

où e est la structure d'Euler combinatoire sur M correspondant à la *Spin*^{c}-structure σ (voir fin du §1.2.3).

EXEMPLE 1.24. Soit (M, σ) une *Spin*^{c}-variété fermée de dimension 3. Nous notons $Q(H_1(M)/TH_1(M))$ le corps des fractions du domaine $\mathbf{Z}[H_1(M)/TH_1(M)]$, où $TH_1(M)$ désigne le sous-groupe de torsion de $H_1(M)$. Considérons la flèche canonique μ définie par la composition :

$$\mathbf{Z}[H_1(M)] \longrightarrow \mathbf{Z}[H_1(M)/TH_1(M)] \hookrightarrow Q(H_1(M)/TH_1(M)).$$

Alors, $\tau^{\mu}(M, \sigma)$ est appelée la *torsion de Milnor* de la *Spin*^{c}-variété (M, σ).

Nous rappelons maintenant le théorème de Franz-Milnor de dualité de la torsion tel qu'il a été raffiné par Turaev.

THÉORÈME 1.25. (Turaev, [**T7**, Th. 20.3], [**T4**, App. B] et [**T3**, App.]) *Soit* (M, σ) *une Spin*^{c}*-variété fermée de dimension* 3. *Soient aussi* $\mathbf{F}$ *un corps commutatif muni d'une involution* $\left(f \longmapsto \overline{f}\right)$ *et* $\mathbf{Z}[H_1(M)] \xrightarrow{\varphi} \mathbf{F}$ *un homomorphisme d'anneaux tels que* $\varphi\left(h^{-1}\right) = \overline{\varphi(h)}$ *pour tout* $h \in H_1(M)$. *Nous avons alors :*

$$\overline{\tau^{\varphi}(M, \sigma)} = \varphi\left(P^{-1}c(\sigma)\right)^{-1} \cdot \tau^{\varphi}(M, \sigma) \in \mathbf{F},$$

où $c(\sigma) \in H^2(M)$ *désigne la classe de Chern de la Spin*^{c}*-structure* σ.

EXEMPLE 1.26. Ce théorème de dualité s'applique à la torsion de Milnor (introduite dans l'Ex. 1.24).

REMARQUE 1.27. Considérons une 3-variété compacte orientée M *de bord une réunion disjointe de tores*. Alors, $\chi(M) = 0$ donc $Eul(M) \neq \varnothing$. Turaev a montré dans [**T4**, §6.6] l'existence d'une bijection canonique et affine :

$$Eul(M) \xrightarrow{\simeq} Spin^c(M, \dot{\cup}\sigma_0)$$

entre les structures d'Euler combinatoires sur M et les $Spin^c$-structures sur M relatives à la structure spinorielle canonique $\dot{\cup}\sigma_0$ de ∂M (voir Ex. 2.16 du Chap. 1). Soient maintenant $\mathbf{F}$ un corps commutatif, $\mathbf{Z}[H_1(M)] \xrightarrow{\varphi} \mathbf{F}$ un homomorphisme d'anneaux, σ une $Spin^c$-structure sur M relative à $\dot{\cup}\sigma_0$ et ω une orientation homologique sur M. Comme dans le cas fermé, est définie alors la *torsion* associée à φ *de la* $Spin^c$*-variété* (M, σ) homologiquement orientée par ω :

$$\tau^{\varphi}(M, \sigma, \omega) \in \mathbf{F}.$$

Cette approche contient en particulier l'étude des polynômes d'Alexander-Conway multivariables ∇_L, où L est une entrelacs orienté et ordonné d'une sphère d'homologie orientée (voir [**T3**, §4] pour une exposition détaillée, ou [**T8**, §8.1]).

Dans la suite, nous continuons à nous consacrer aux torsions des $Spin^c$-variétés fermées de dimension 3. Cependant, nos prochains résultats s'étendent sans difficultés à ce cas à bord *particulier* que nous venons d'esquisser.

2. Variation de la torsion lors d'un twist

Considérons une 3-variété fermée orientée :

$$M = M_1 \cup_f M_2,$$

obtenue en recollant deux 3-variétés orientées M_1 et M_2 le long de leur bord, par un difféomorphisme positif $-\partial M_2 \xrightarrow{f} \partial M_1$. Nous supposons que $\partial M_1 \cong -\partial M_2$ *est connexe.* Soit aussi $h \in \mathcal{T}(\partial M_2)$ un élément du groupe de Torelli de la surface ∂M_2. Formons :

$$M' = M_1 \cup_{f \circ h} M_2.$$

Ainsi, la variété M' est obtenue en twistant M le long de la surface ∂M_2 par h. Au Chapitre 1, nous avons montré l'existence, dans cette situation, d'une bijection canonique :

$$Spin^c(M) \underset{\simeq}{\xrightarrow{\Omega}} Spin^c(M'),$$

qui est affine au-dessus de $P\Phi P^{-1}$ où

$$H_1(M) \underset{\simeq}{\xrightarrow{\Phi}} H_1(M')$$

est l'isomorphisme d'homologie donné par le théorème de Mayer-Vietoris. Soit un homomorphisme d'anneaux :

$$\mathbf{Z}[H_1(M)] \xrightarrow{\varphi} \mathbf{F}$$

à valeurs dans un corps commutatif $\mathbf{F}$ et soit $\alpha \in Spin^c(M)$. Nous notons alors φ' l'homomorphisme d'anneaux $\varphi\Phi^{-1}$ et α' la $Spin^c$-structure $\Omega(\alpha)$ sur M'.

Le but de cette section est de comparer les torsions de Reidemeister-Turaev :

$$\tau^{\varphi}(M, \alpha) \in \mathbf{F} \quad \text{et} \quad \tau^{\varphi'}(M', \alpha') \in \mathbf{F},$$

c'est-à-dire de trouver un scalaire $C \in \mathbf{F}$ tel que

$$\tau^{\varphi'}(M', \alpha') = C \cdot \tau^{\varphi}(M, \alpha).$$

En général, ce scalaire ne sera identifié qu'à un facteur multiplicatif près dans $\varphi(H_1(M))$. Cette indétermination sera levée sous certaines hypothèses sur le corps $\mathbf{F}$ et l'homomorphisme φ.

L'ingrédient pour cette étude est une formule de Mayer-Vietoris pour la torsion, que nous présentons dès maintenant.

2.1. Formule contingente de Mayer-Vietoris pour la torsion. Nous considérons dans ce paragraphe un CW-complexe[4] K de dimension m, de support $|K| = X$ et de caractéristique d'Euler $\chi(X) = 0$. Nous fixons aussi une structure d'Euler combinatoire $e \in Eul(K)$, une orientation homologique ω de X et :

$$\mathbf{Z}\left[H_1(X)\right] \xrightarrow{\varphi} \mathbf{F}$$

un homomorphisme d'anneaux à valeurs dans un corps commutatif $\mathbf{F}$.

Supposons que K est décomposé en deux sous-complexes :

$$K = K_1 \ \cup \ K_2,$$

où K_1, K_2 et $K_1 \cap K_2$ sont connexes (nous notons X_1 et X_2 les supports respectifs de K_1 et K_2). On se propose ici de mettre en application la multiplicativité des torsions, vue en §1.1.2, pour obtenir une formule du type "Mayer-Vietoris" exprimant la torsion :

$$\tau^{\varphi}(K; e, \omega)$$

en fonction de torsions associées aux CW-complexes K_1, K_2 et $K_1 \cap K_2$. Les caractéristiques d'Euler de ces derniers étant supposées *quelconques*, les torsions mises en jeu seront dépendantes de choix intermédiaires. Pour ces raisons, notre formule de Mayer-Vietoris sera hautement non-canonique, et sera de ce fait qualifiée de *contingente.*

Nous décrivons maintenant les différents choix permettant d'énoncer cette formule de Mayer-Vietoris. Pour cela, notons les inclusions comme suit :

$$\begin{tikzcd}
& X_1 \arrow[dr, "i_1"] & \\
X_1 \cap X_2 \arrow[ur, "j_1"] \arrow[rr, "l"] \arrow[dr, "j_2"'] & & X \\
& X_2 \arrow[ur, "i_2"'] &
\end{tikzcd}$$

Nous commençons par fixer un point $\star$ sur $X_1 \cap X_2$, qui servira de point base pour $X_1 \cap X_2$, X_1, X_2 et X.

Puis, nous décrivons *comment bien choisir des orientations homologiques.* Soient ϖ, ω_1 et ω_2 des orientations homologiques de respectivement $X_1 \cap X_2$, X_1 et X_2. Piochons pour tout $i \in \{0, \dots, m\}$ une base w_i de $H_i(X; \mathbf{R})$ telles que la base $w = (w_0, w_1, \dots)$ de $H_*(X; \mathbf{R}) = \oplus_{i \geq 0} H_i(X; \mathbf{R})$ représente l'orientation homologique ω. Similairement, soient pour tout $i = 0, \dots, m$ des bases $\overline{w}_i$, $w_{1,i}$ et $w_{2,i}$ telles que $\overline{w} = (\overline{w}_0, \overline{w}_1, \dots)$, $w_1 = (w_{1,0}, w_{1,1}, \dots)$ et $w_2 = (w_{2,0}, w_{2,1}, \dots)$ représentent respectivement les orientations homologiques ϖ, ω_1 et ω_2. Considérons le $\mathbf{R}$-complexe acyclique :

$$\cdots \longrightarrow H_i(X_1 \cap X_2; \mathbf{R}) \xrightarrow{(j_{1,*}, j_{2,*})} H_i(X_1; \mathbf{R}) \oplus H_i(X_2; \mathbf{R}) \xrightarrow{i_{1,*} \oplus (-i_{2,*})} H_i(X; \mathbf{R}) \cdots$$

associé à la suite longue de Mayer-Vietoris de la réunion $X = X_1 \cup X_2$. Le signe de la torsion de ce $\mathbf{R}$-complexe basé par $(\dots, \overline{w}_i, (w_{1,i}, w_{2,i}), w_i, \dots)$ et homologiquement basé par $\varnothing$ ne dépend que de ϖ, ω_1, ω_2 et ω. Les orientations homologiques ϖ, ω_1 et ω_2 *sont bien choisies* lorsque ce signe-là est positif.

[4]On rappelle que dans ce chapitre, par convention, les CW-complexes sont finis et connexes.

Nous décrivons ensuite *comment bien choisir des bases homologiques.* Soient $\widehat{l}$, $\widehat{j_k}$ et $\widehat{i_k}$ les relèvements respectifs de l, j_k et i_k aux revêtements abéliens maximaux définis par le choix du point base $\star$ (voir fin du §1.2.4) . Bien sûr, tous ces relèvements sont équivariants au-dessus des applications correspondantes pour les espaces bases. Elles induisent donc des applications de chaînes entre les complexes :

$$C_*^{\varphi}(K,\star),\ C_*^{\varphi o l_*}(K_1 \cap K_2,\star) \quad \text{et :} \quad C_*^{\varphi \circ i_{k,*}}(K_k,\star),\ k=1,2.$$

D'où la suite exacte courte de $\mathbf{F}$-complexes :

$$0 \longrightarrow C_*^{\varphi o l_*}(K_1 \cap K_2,\star) \xrightarrow{\left(\widehat{j_1},\widehat{j_2}\right)} C_*^{\varphi \circ i_{1,*}}(K_1,\star) \oplus C_*^{\varphi \circ i_{2,*}}(K_2,\star) \xrightarrow{\widehat{i_1}\oplus\left(-\widehat{i_2}\right)} C_*^{\varphi}(K,\star) \longrightarrow 0.$$

De la suite longue d'homologie associée, on déduit que $H_*^{\varphi}(X,\star)=0$ si et seulement si :

$$H_*^{\varphi l_*}(X_1 \cap X_2,\star) \xrightarrow{(j_{1,*},j_{2,*})} H_*^{\varphi i_{1,*}}(X_1,\star) \oplus H_*^{\varphi i_{2,*}}(X_2,\star)$$

est un isomorphisme. Lorsque ce cas-là se présente, *un bon choix de bases homologiques* c_i de $H_i^{\varphi l_*}(X_1 \cap X_2,\star)$, $b_{1,i}$ de $H_i^{\varphi i_{1,*}}(X_1,\star)$ et $b_{2,i}$ de $H_i^{\varphi i_{2,*}}(X_2,\star)$ est un choix tel que $(j_{1,*},j_{2,*})(c_i)=(b_{1,i},b_{2,i})$. On note alors : $c=(c_0,c_1,\dots)$, $b_1=(b_{1,0},b_{1,1},\dots)$ et $b_2=(b_{2,0},b_{2,1},\dots)$.

Nous décrivons enfin *comment bien choisir des familles fondamentales de cellules.* Soit $\mathfrak{e} \in E(K,\star)$ correspondant à la structure d'Euler combinatoire e. *Un bon choix de familles fondamentales de cellules* $\widehat{f}$, $\widehat{e}_1$ et $\widehat{e}_2$ pour les revêtements abéliens maximaux de respectivement $(K_1 \cap K_2,\star)$, $(K_1,\star)$ et $(K_2,\star)$, est un choix tel que[5] :

– pour $k=1$ ou 2 et pour toute cellule σ de $K_1 \cap K_2$, on ait :

$$\widehat{e}_k\left(j_k(\sigma)\right)=\widehat{j_k}\left(\widehat{f}(\sigma)\right);$$

– $\widehat{e} := \widehat{i_1}\left(\widehat{e}_1 \setminus \widehat{j_1}\left(\widehat{f}\right)\right) \dot\cup\, \widehat{l}\left(\widehat{f}\right) \dot\cup\, \widehat{i_2}\left(\widehat{e}_2 \setminus \widehat{j_2}\left(\widehat{f}\right)\right)$ représente $\mathfrak{e}$.

Nous noterons alors $\mathfrak{f} \in E(K_1 \cap K_2,\star)$, $\mathfrak{e}_1 \in E(K_1,\star)$ et $\mathfrak{e}_2 \in E(K_2,\star)$ les classes respectives de $\widehat{f}$, $\widehat{e}_1$ et $\widehat{e}_2$.

LEMME 2.1. *Sous les hypothèses et notations précédentes, la torsion $\tau^{\varphi}(K;e,\omega)$ est non-nulle si et seulement si :*

$$H_*^{\varphi l_*}(X_1 \cap X_2,\star) \xrightarrow{(j_{1,*},j_{2,*})} H_*^{\varphi i_{1,*}}(X_1,\star) \oplus H_*^{\varphi i_{2,*}}(X_2,\star)$$

est un isomorphisme[6]. Dans ce cas, et lorsque des bons choix ont été faits comme décrits précédemment, la formule suivante a lieu :

$$\tau^{\varphi}(K;e,\omega)=\epsilon \cdot \frac{\check{\tau}^{\varphi \circ i_{1,*}}(K_1,\star;\mathfrak{e}_1,b_1,\omega_1)\cdot \check{\tau}^{\varphi \circ i_{2,*}}(K_2,\star;\mathfrak{e}_2,b_2,\omega_2)}{\check{\tau}^{\varphi o l_*}(K_1 \cap K_2,\star;\mathfrak{f},c,\varpi)}, \tag{2.1}$$

où ϵ est un signe explicite qui ne dépend que des espaces vectoriels gradués : $H_*(X_1 \cap X_2;\mathbf{R})$, $H_*(X_1;\mathbf{R})$, $H_*(X_2;\mathbf{R})$, $H_*(X;\mathbf{R})$, $H_*^{\varphi i_{1,*}}(X_1,\star)$ *et* $H_*^{\varphi i_{2,*}}(X_2,\star)$, *à isomorphisme près.*

[5] Un tel choix est possible. En effet, il existe évidemment un triplet de familles fondamentales de cellules $\left(\widehat{f},\widehat{e}_1,\widehat{e}_2\right)$ satisfaisant à la première condition énoncée ; puis, par action libre transitive de $H_1(X)$ sur $E(K,\star)$ et par surjectivité de $H_1(X_1)\oplus H_1(X_2) \xrightarrow{(i_{1,*},i_{2,*})} H_1(X)$, on trouve alors, en modifiant $\widehat{e}_1$ et $\widehat{e}_2$, un triplet $\left(\widehat{f},\widehat{e}_1,\widehat{e}_2\right)$ tel que la classe de $\widehat{e}$, définie par la deuxième condition, soit $\mathfrak{e}$.

[6] Cette condition est indépendante du choix du point base $\star$.

Démonstration. La première assertion découle du fait que, par définition, $\tau^{\varphi}(K;e,\omega)\neq 0$ si et seulement si $H_*^{\varphi}(X,\star)=0$, et de la suite de Mayer-Vietoris pour l'homologie à coefficients twistés. Plaçons-nous dans ce cas pour la suite.

Soient $w=(w_0,\dots,w_m)$, $\overline{w}=(\overline{w}_0,\dots,\overline{w}_m)$, $w_1=(w_{1,0},\dots,w_{1,m})$ et $w_2=(w_{2,0},\dots,w_{2,m})$ des bases respectives de $H_*(X;\mathbf{R})$, $H_*(X_1\cap X_2;\mathbf{R})$, $H_*(X_1;\mathbf{R})$ et $H_*(X_2;\mathbf{R})$ représentant respectivement les bases homologiques ω, ϖ, ω_1 et ω_2. Soit aussi oo un choix d'*o*rientations et d'*o*rdre des cellules de K (induisant aussi ce même type de choix pour les complexes $K_1\cap K_2$, K_1, K_2). Rappelons-nous alors que $\tau^{\varphi}(K;e,\omega)$ vaut :

$$\check{\tau}^{\varphi}\left(K,\star;\mathfrak{e},\varnothing,\omega\right) \quad = \quad \operatorname{sgn}\left(\check{\tau}\left(C_*(K;\mathbf{R});oo,w\right)\right)\cdot\check{\tau}\left(C_*^{\varphi}\left(K,\star\right);\widehat{e}_{oo},\varnothing\right),$$

que pour $k=1$ ou 2 :

$$\check{\tau}^{\varphi i_{k,*}}\left(K_k,\star;\mathfrak{e}_k,b_k,\omega_k\right)= \\ \operatorname{sgn}\left(\check{\tau}\left(C_*(K_k;\mathbf{R});oo,w_k\right)\right)\cdot\check{\tau}\left(C_*^{\varphi i_{k,*}}\left(K_k,\star\right);\widehat{e}_{k,oo},b_k\right),$$

et enfin que :

$$\check{\tau}^{\varphi l_*}\left(K_1\cap K_2,\star;\mathfrak{f},c,\varpi\right)= \\ \operatorname{sgn}\left(\check{\tau}\left(C_*(K_1\cap K_2;\mathbf{R});oo,\overline{w}\right)\right)\cdot\check{\tau}\left(C_*^{\varphi l_*}\left(K_1\cap K_2,\star\right);\widehat{f}_{oo},c\right).$$

On applique la formule (1.5) à la s.e.c. de $\mathbf{F}$-complexes :

$$0\longrightarrow C_*^{\varphi\circ l_*}\left(K_1\cap K_2,\star\right)\xrightarrow{(\widehat{j_1},\widehat{j_2})} C_*^{\varphi\circ i_{1,*}}\left(K_1,\star\right)\oplus C_*^{\varphi\circ i_{2,*}}\left(K_2,\star\right) \\ \xrightarrow{\widehat{i_1}\oplus(-\widehat{i_2})} C_*^{\varphi}\left(K,\star\right)\longrightarrow 0.$$

et nous obtenons :

$$\check{\tau}\left(C_*^{\varphi i_{1*}}\left(K_1,\star\right)\oplus C_*^{\varphi i_{2*}}\left(K_2,\star\right);\left(\widehat{e}_{1,oo},\widehat{e}_{2,oo}\right),\left(b_1,b_2\right)\right)= \\ (-1)^{\mu+\nu}\cdot\check{\tau}\left(C_*^{\varphi l_*}\left(K_1\cap K_2,\star\right);\widehat{f}_{oo},c\right)\cdot\check{\tau}\left(C_*^{\varphi}\left(K,\star\right);\widehat{e}_{oo},\varnothing\right), \tag{2.2}$$

où $(-1)^{\mu}$ est un signe provenant du fait que les bases $\widehat{f}_{oo}$, $(\widehat{e}_{1,oo},\widehat{e}_{2,oo})$ et $\widehat{e}$ des $\mathbf{F}$-complexes mis en jeu, ne sont éventuellement pas compatibles mais, telles que nous les avons bien choisies, le sont certainement au signe près. Et :

$$\nu=\sum_{i=0}^{m}\alpha_i\left(C_*^{\varphi}\left(K,*\right)\right)\cdot\alpha_{i-1}\left(C_*^{\varphi l_*}\left(K_1\cap K_2,*\right)\right)\in\mathbf{Z}_2.$$

Notons que nous avons utilisé le fait que le bases homologiques c, b_1 et b_2 étaient bien choisies.

D'après la formule (1.4) appliquée à la s.e.c. de $\mathbf{R}$-complexes :

$$0\longrightarrow C_*\left(K_1\cap K_2;\mathbf{R}\right)\xrightarrow{(j_1,j_2)} C_*\left(K_1;\mathbf{R}\right)\oplus C_*\left(K_2;\mathbf{R}\right)\xrightarrow{(i_1,-i_2)} C_*\left(K;\mathbf{R}\right)\longrightarrow 0$$

nous avons :

$$\operatorname{sgn}\left(\check{\tau}\left(C_*(K_1;\mathbf{R})\oplus C_*(K_2;\mathbf{R});oo,(w_1,w_2)\right)\right)= \\ (-1)^{\mu'+\nu'+\theta'}\cdot 1\cdot\operatorname{sgn}\left(\check{\tau}\left(C_*(K_1\cap K_2;\mathbf{R});oo,\overline{w}\right)\right)\cdot\operatorname{sgn}\left(\check{\tau}\left(C_*(K;\mathbf{R});oo,w\right)\right). \tag{2.3}$$

Ici, le signe $(-1)^{\theta'}$ peut-être explicitement écrit comme fonction des $\mathbf{R}$-espaces vectoriels gradués $H_*(X_1\cap X_2;\mathbf{R})$, $H_*(X_1;\mathbf{R})$, $H_*(X_2;\mathbf{R})$ et $H_*(X;\mathbf{R})$. Le bon choix des orientations homologiques ϖ, ω_1 et ω_2 conduit à la constante 1 dans cette formule. Le signe $(-1)^{\mu'}$ provient du fait que les bases des $\mathbf{R}$-complexes concernés ne sont peut-être pas compatibles entre-elles. Enfin :

$$\nu'=\sum_{i=0}^{m}\alpha_i\left(C_*\left(K;\mathbf{R}\right)\right)\cdot\alpha_{i-1}\left(C_*\left(K_1\cap K_2;\mathbf{R}\right)\right)\in\mathbf{Z}_2.$$

Remarquons les égalités de signes : $(-1)^{\mu}=(-1)^{\mu'}$ et $(-1)^{\nu}=(-1)^{\nu'}$.

La formule (1.6) pour la somme directe $C_*^{\varphi i_{1*}}(K_1,\star) \oplus C_*^{\varphi i_{2*}}(K_2,\star)$ nous donne :

$$\check{\tau}\left(C_*^{\varphi i_{1*}}(K_1,\star) \oplus C_*^{\varphi i_{2*}}(K_2,\star);(\widehat{e}_{1,oo},\widehat{e}_{2,oo}),(b_1,b_2)\right) = \tag{2.4}$$

$$(-1)^{\rho}\cdot\check{\tau}\left(C_*^{\varphi i_{1*}}(K_1,\star);\widehat{e}_{1,oo},b_1\right)\cdot\check{\tau}\left(C_*^{\varphi i_{2*}}(K_2,\star);\widehat{e}_{2,oo},b_2\right),$$

où $\rho \in \mathbf{Z}_2$ est défini par :

$$\rho = \left(\sum_{i=0}^{m}\alpha_i\left(C_*^{\varphi i_{2*}}(K_2,\star)\right)\alpha_{i-1}\left(C_*^{\varphi i_{1*}}(K_1,\star)\right)\right) +$$
$$\left(\sum_{i=0}^{m}\beta_i\left(C_*^{\varphi i_{2*}}(K_2,\star)\right)\beta_{i-1}\left(C_*^{\varphi i_{1*}}(K_1,\star)\right)\right).$$

De même, nous pouvons appliquer la formule (1.6) à la somme directe $C_*(K_1;\mathbf{R}) \oplus C_*(K_2;\mathbf{R})$ pour obtenir :

$$\operatorname{sgn}\left(\check{\tau}\left(C_*(K_1;\mathbf{R}) \oplus C_*(K_2;\mathbf{R});oo,(w_1,w_2)\right)\right) = \tag{2.5}$$

$$(-1)^{\rho'}\cdot\operatorname{sgn}\left(\check{\tau}\left(C_*(K_1;\mathbf{R});oo,w_1\right)\right)\cdot\operatorname{sgn}\left(\check{\tau}\left(C_*(K_2;\mathbf{R});oo,w_2\right)\right).$$

où $\rho' \in \mathbf{Z}_2$ est défini par :

$$\rho' = \left(\sum_{i=0}^{m}\alpha_i\left(C_*\left(K_2;\mathbf{R}\right)\right)\alpha_{i-1}\left(C_*\left(K_1;\mathbf{R}\right)\right)\right) +$$
$$\left(\sum_{i=0}^{m}\beta_i\left(C_*\left(K_2;\mathbf{R}\right)\right)\beta_{i-1}\left(C_*\left(K_1;\mathbf{R}\right)\right)\right).$$

Nous remarquons que le signe $(-1)^{\rho+\rho'}$ peut être explicité en fonction des espaces vectoriels gradués annoncés par le lemme. La formule (2.1) découle alors de (2.2), (2.3), (2.4) et (2.5). $\square$

2.2. Homologie à coefficients twistés d'une surface. Soit Σ une surface compacte orientable de genre g avec 0 ou 1 composante de bord :

$$\Sigma = \Sigma_g \quad \text{ou} \quad \Sigma_{g,1},$$

et soit un homomorphisme d'anneaux à valeurs dans un corps commutatif $\mathbf{F}$:

$$\mathbf{Z}\left[H_1(\Sigma)\right] \xrightarrow{\varphi} \mathbf{F}.$$

Nous nous intéressons maintenant au calcul de $H_*^{\varphi}(\Sigma,\star)$, où $\star \in \Sigma$.

Dans le cas particulier où $\varphi(x) = 1 \in \mathbf{F}$ pour tout $x \in H_1(\Sigma)$, nous avons bien sûr :

$$H_*^{\varphi}(\Sigma,\star) \simeq H_*(\Sigma;\mathbf{F}) \quad \text{(isomorphisme canonique)}.$$

Ce cas-là étant parfaitement entendu, nous l'excluons dans la suite.

2.2.1. *Calcul de* $H_*^{\varphi}(\Sigma_{g,1},\star)$. Soit $\star \in \partial\Sigma_{g,1}$ et $x_1,y_1,\ldots,x_g,y_g$ les courbes simples fermées orientées basées en $\star$ qui sont dessinées sur la Figure 2.1. Nous notons $n = 2g$ et $z_1 = x_1,\ z_2 = y_1,\ldots,\ z_{n-1} = x_g,\ z_n = y_g$. Soient aussi $H = H_1(\Sigma_{g,1})$, $\pi = \pi_1(\Sigma_{g,1},\star)$ et $\pi \xrightarrow{a} H$ l'homomorphisme de Hurewicz.

Le groupe π étant librement engendré par $z_1,\ldots,z_n$, nous pouvons lui appliquer le calcul différentiel libre de Fox (nous renvoyons le lecteur à [**Bi**, Chap. 3] pour une introduction à ce formalisme). Soit donc pour tout $i \in \{1,\ldots,n\}$:

$$\mathbf{Z}[\pi] \xrightarrow{\frac{\partial}{\partial z_i}} \mathbf{Z}[\pi]$$

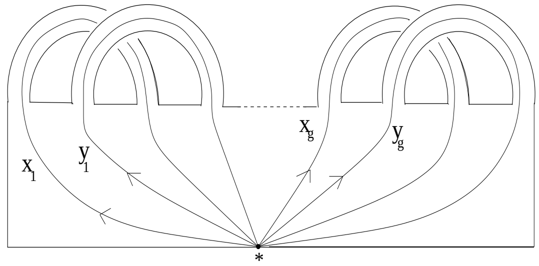

FIG. 2.1 – Les courbes $(x_i, y_i)_{i=1}^{g}$ sur $\Sigma_{g,1}$

la $i^{\text{ème}}$ *dérivation partielle libre*. Si ρ est un endomorphisme du groupe π, nous lui associons *sa matrice Jacobienne* :

$$J(\rho) = \left(\frac{\partial \rho(z_j)}{\partial z_i}\right)_{i,j} \in \mathcal{M}_n\left(\mathbf{Z}[\pi]\right)$$

et la réduction abélienne de cette dernière :

$$aJ(\rho) = \left(a\left(\frac{\partial \rho(z_j)}{\partial z_i}\right)\right)_{i,j} \in \mathcal{M}_n\left(\mathbf{Z}[H]\right).$$

Nous notons $\mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle$ le $\mathbf{F}$-espace vectoriel formel de base $(e_{z_1}, \ldots, e_{z_n})$, et pour tout $x \in \pi$ nous posons :

$$e_x = \sum_{i=1}^{n} \varphi a\left(\frac{\partial x}{\partial z_i}\right) \cdot e_{z_i} \in \mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle.$$

LEMME 2.2. *a) Nous avons :*

$$\begin{cases} H_r^{\varphi}\left(\Sigma_{g,1}, \star\right) = 0 & \forall r \neq 1, \\ H_1^{\varphi}\left(\Sigma_{g,1}, \star\right) \simeq \mathbf{F}^{2g-1}. \end{cases}$$

b) Précisément, si m désigne la forme linéaire

$$\begin{cases} \mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle & \xrightarrow{m} & \mathbf{F} \\ \sum_{i=1}^{n} f_i \cdot e_{z_i} & \longmapsto & \sum_{i=1}^{n} f_i \cdot (\varphi(z_i) - 1), \end{cases}$$

alors le choix de la base $(z_1, \ldots, z_n)$ de π induit une identification :

$$H_1^{\varphi}\left(\Sigma_{g,1}, \star\right) \xrightarrow[\simeq]{} Ker(m).$$

c) Soit $\Sigma_{g,1} \xrightarrow[\simeq]{h} \Sigma_{g,1}$ un difféomorphisme fixant point par point $\partial\Sigma_{g,1}$ et induisant l'identité au niveau de H. Alors le diagramme suivant commute :

$$\begin{tikzcd} H_1^{\varphi}\left(\Sigma_{g,1}, \star\right) \arrow[r, "\simeq"'] \arrow[d, "h_*"'] & Ker(m) \arrow[r, hook] & \mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle \arrow[d, "h_*"] \\ H_1^{\varphi}\left(\Sigma_{g,1}, \star\right) \arrow[r, "\simeq"'] & Ker(m) \arrow[r, hook] & \mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle. \end{tikzcd}$$

Ici, la flèche verticale h_* de droite est l'automorphisme de $\mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle$ dont la matrice dans la base $(e_{z_1}, \ldots, e_{z_n})$ est $\varphi aJ(h_*) \in GL_n(\mathbf{F})$.

d) Soit L un espace topologique et soit $\mathbf{Z}\left[H_1(L)\right] \xrightarrow{\psi} \mathbf{F}$ un homomorphisme *d'anneaux. Soit aussi une application* $\Sigma_{g,1} \xrightarrow{f} L$ *telle que* $\varphi = \psi \circ f_*$*. Si* $x \in \pi$ *vérifie* $f_*(x) = 1 \in \pi_1(L, f(\star))$*, alors :*

$$H_1^{\varphi}(\Sigma_{g,1}, \star) \xrightarrow{f_*} H_1^{\psi}(L, f(\star))$$

s'annule sur le vecteur e_x*.*

Démonstration. On note $X \subset \Sigma_{g,1}$ le bouquet de n cercles suivant :

$$X = \bigcup_{i=1}^{n} z_i$$

auquel se rétracte par déformation la surface $\Sigma_{g,1}$. Le calcul de $H_*^{\varphi}(\Sigma_{g,1}, \star)$ est donc ramené à celui de $H_*^{\varphi}(X, \star)$ (la composée de φ avec $H_1(X) \xrightarrow{\simeq} H_1(\Sigma_{g,1}) = H$ est encore notée φ).

Soit $\left(\widehat{X}, p\right)$ le revêtement abélien maximal de $(X, \star)$. La décomposition cellulaire K de X avec $\star$ pour unique 0-cellule et $z_1, \ldots, z_n$ pour 1-cellules se relève en une décomposition cellulaire $\widehat{K}$ de $\widehat{X}$. Soit $\widehat{z}_i$ l'unique relèvement de z_i débutant en $\widehat{\star}$ (point base préféré de $\widehat{X}$ défini à la fin du §1.2.4). Le complexe :

$$C_*^{\varphi}(K, \star) = \mathbf{F} \otimes_{\mathbf{Z}[H]} C_*\left(\widehat{K}\right)$$

est concentré en degrés 1 et 0 donc, en particulier :

$$\forall r \neq 0, 1, \quad H_r^{\varphi}(X, \star) = 0.$$

Le $\mathbf{F}$-espace vectoriel $C_1^{\varphi}(K, \star)$ admet $(e_{z_1}, \ldots, e_{z_n})$ pour base où $e_{z_i} := 1 \otimes \widehat{z}_i$; le $\mathbf{F}$-espace vectoriel $C_0^{\varphi}(K, \star)$ est de dimension 1 engendré par $e_\star := 1 \otimes \widehat{\star}$. L'opérateur de bord $C_1^{\varphi}(K, \star) \xrightarrow{\partial_0} C_0^{\varphi}(K, \star)$ s'écrit alors :

$$\partial_0\left(\sum_{i=1}^{n} f_i \cdot e_{z_i}\right) = \underbrace{\left(\sum_{i=1}^{n} f_i \cdot (\varphi(z_i) - 1)\right)}_{\in \mathbf{F}} \cdot e_\star$$

Soit $i_0 \in \{1, \ldots, n\}$ tel que $\varphi(z_{i_0}) \neq 1 \in \mathbf{F}$ alors $\partial_0\left(1/\left(\varphi(z_{i_0}) - 1\right) \cdot e_{z_{i_0}}\right) = e_\star$, d'où :

$$H_0^{\varphi}(X, \star) = 0.$$

De plus, $H_1^{\varphi}(X, \star) = Ker(\partial_0)$, et donc en particulier :

$$H_1^{\varphi}(X, \star) \simeq \mathbf{F}^{n-1},$$

ce qui prouve les points a) et b) du lemme.

Nous prouvons maintenant l'assertion c). Soit pour cela $X \xrightarrow{h} X$ correspondant à $\Sigma_{g,1} \xrightarrow{h} \Sigma_{g,1}$ par rétraction, et soit $\widehat{X} \xrightarrow{\widehat{h}} \widehat{X}$ son relèvement (voir fin du §1.2.4). Cette application $\widehat{h}$ induit une flèche $C_1\left(\widehat{K}\right) \xrightarrow{\widehat{h}} C_1\left(\widehat{K}\right)$ puis, h préservant l'homologie, une flèche :

$$C_1^{\varphi}(K, \star) \xrightarrow{\widehat{h}} C_1^{\varphi}(K, \star).$$

On souhaite démontrer que la matrice de cet endomorphisme dans la base $(e_{z_i})_{i=1}^{n}$ est égale à $\varphi a J(h_*) \in GL_n(\mathbf{F})$.
Nous rappelons que, d'une façon générale, si γ est un lacet dans X orienté et basé

en $\star$ représentant $[\gamma] \in \pi$, et si $\widehat{\gamma}$ est l'unique relèvement de γ à $\widehat{X}$ débutant en $\widehat{\star}$, alors :

$$C_1\left(\widehat{K}\right) \ni \widehat{\gamma} = \sum_{i=1}^{n} \underbrace{a\left(\frac{\partial[\gamma]}{\partial z_i}\right)}_{\in \mathbf{Z}[H]} \cdot \widehat{z_i}. \tag{2.6}$$

Or, ayant :

$$\widehat{h}\left(e_{z_j}\right) = 1 \otimes \widehat{h}\left(\widehat{z_j}\right) \in C_1^{\varphi}(K, \star)$$

et $\widehat{h}\left(\widehat{z_j}\right)$ étant l'unique relèvement de $h_*(z_j) \in \pi$ débutant en $\widehat{\star}$, la formule (2.6) appliquée à un lacet γ représentant $h_*(z_j)$ implique que :

$$C_1^{\varphi}(K, \star) \ni \widehat{h}\left(e_{z_j}\right) = \sum_{i=1}^{n} \varphi a\left(\frac{\partial h_*(z_j)}{\partial z_i}\right) \cdot e_{z_i}.$$

Nous démontrons maintenant le point d). Soit γ un lacet basé en $\star$ représentant x et soit $\widehat{\gamma}$ son relèvement débutant en $\widehat{\star}$. Alors, dans $\mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle = C_1^{\varphi}(K, \star)$, e_x est égal d'après (2.6) à $1 \otimes \widehat{\gamma}$. Ainsi, $f_*(e_x) \in H_1^{\psi}(L, f(\star))$ est représenté par $1 \otimes \widehat{f}(\widehat{\gamma}) \in C_1^{\psi}(L, f(\star))$. Or, la projection de $\widehat{f}(\widehat{\gamma})$ sur L est nul-homotope par hypothèse, donc le chemin $\widehat{f}(\widehat{\gamma})$ est un lacet nul-homotope du revêtement abélien maximal de $(L, f(\star))$, si bien que $f_*(e_x) = 0$. $\square$

2.2.2. *Calcul de $H_*^{\varphi}(\Sigma_g, \star)$.* Nous fixons une inclusion $\Sigma_{g,1} \subset \Sigma_g$, notons $D \subset \Sigma_g$ le disque complémentaire et maintenons les notations du §2.2.1 relatives à la surface $\Sigma_{g,1}$.

LEMME 2.3. *a) Nous avons :*

$$\begin{cases} H_r^{\varphi}(\Sigma_g, \star) = 0 & \forall r \neq 1, \\ H_1^{\varphi}(\Sigma_g, \star) \simeq \mathbf{F}^{2g-2}. \end{cases}$$

b) Précisément, le choix de l'inclusion $\Sigma_{g,1} \subset \Sigma_g$ et le choix de la base $(z_1, \ldots, z_n)$ de π induisent une identification :

$$H_1^{\varphi}(\Sigma_g, \star) \xrightarrow[\simeq]{} \frac{Ker(m)}{\langle\omega\rangle},$$

où $Ker(m)$ est l'hyperplan défini par le Lemme 2.2 et où :

$$\omega = \sum_{i=1}^{g} \varphi(y_i)^{-1} \cdot \Big((\varphi(y_i) - 1) \cdot e_{x_i} + (1 - \varphi(x_i)) \cdot e_{y_i}\Big) \in Ker(m).$$

c) Soit $\Sigma_{g,1} \xrightarrow[\simeq]{h} \Sigma_{g,1}$ un difféomorphisme fixant point par point $\partial\Sigma_{g,1}$ et induisant l'identité au niveau de H. On note encore $\Sigma_g \xrightarrow[\simeq]{h} \Sigma_g$ l'extension de h par l'identité de D. Alors le diagramme suivant commute :

$$\begin{tikzcd}
H_1^{\varphi}(\Sigma_g, \star) \arrow[r, "\simeq"'] \arrow[d, "h_*"'] & \frac{Ker(m)}{\langle\omega\rangle} & Ker(m) \arrow[l, two heads] \arrow[r, hook] & \mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle \arrow[d, "h_*"] \\
H_1^{\varphi}(\Sigma_g, \star) \arrow[r, "\simeq"'] & \frac{Ker(m)}{\langle\omega\rangle} & Ker(m) \arrow[l, two heads] \arrow[r, hook] & \mathbf{F}\langle e_{z_1}, \ldots, e_{z_n}\rangle.
\end{tikzcd}$$

DÉMONSTRATION. Soit $\left(\widehat{\Sigma_{g,1}}, p\right)$ le revêtement abélien maximal de $(\Sigma_{g,1}, \star)$ et soit $\left(\widehat{\Sigma_g}, p\right)$ celui de $(\Sigma_g, \star)$. Soit K une décomposition cellulaire de $\Sigma_{g,1}$; $L :=$

$K \cup D$ est alors une décomposition cellulaire de Σ_g. La suite exacte courte de $\mathbf{Z}[H]$-complexes libres :

$$0 \longrightarrow C_*\left(\widehat{K}\right) \longrightarrow C_*\left(\widehat{L}\right) \longrightarrow C_*\left(\widehat{L},\widehat{K}\right) \longrightarrow 0$$

induit par application du foncteur $\mathbf{F} \otimes_{\mathbf{Z}[H]} -$ une s.e.c. de $\mathbf{F}$-complexes qui à son tour induit une suite longue d'homologie :

$$\cdots \longrightarrow H_r^{\varphi}\left(\Sigma_{g,1},\star\right) \longrightarrow H_r^{\varphi}\left(\Sigma_g,\star\right) \longrightarrow H_r\left(\mathbf{F}\otimes_{\mathbf{Z}[H]} C_*\left(\widehat{L},\widehat{K}\right)\right) \longrightarrow \cdots$$

Or, le complexe $C_*\left(\widehat{L},\widehat{K}\right)$ est concentré en dimension 2 et $C_2\left(\widehat{L},\widehat{K}\right)$ est $\mathbf{Z}[H]$-libre engendré par $\widehat{D}$, qui est l'unique relèvement de D touchant le point $\widehat{\star}$. Nous déduisons de la suite longue d'homologie que :

$$H_1^{\varphi}\left(\Sigma_g,\star\right) \xleftarrow{\simeq} \frac{H_1^{\varphi}\left(\Sigma_{g,1},\star\right)}{\left\langle\left[1\otimes\partial\widehat{D}\right]\right\rangle}.$$

Le Lemme 2.3 découlera donc du Lemme 2.2 une fois démontrée l'égalité :

$$\left[1\otimes\partial\widehat{D}\right] = \left[\sum_{i=1}^{g}\varphi(y_i)^{-1}\Big((\varphi(y_i)-1)\otimes\widehat{x_i} + (1-\varphi(x_i))\otimes\widehat{y_i}\Big)\right] = \omega. \tag{2.7}$$

Lorsque D est convenablement orienté, ∂D représente :

$$\prod_{j=1}^{g}\left[x_j, y_j^{-1}\right] \in \pi = \pi_1\left(\Sigma_{g,1},\star\right).$$

Ainsi, d'après l'équation (2.6) rappelée durant la preuve du Lemme 2.2 :

$$H_1^{\varphi}\left(\Sigma_{g,1},\star\right) \ni \left[1\otimes\partial\widehat{D}\right] = \left[\sum_{i=1}^{n} 1\otimes\left(a\frac{\partial}{\partial z_i}\left(\prod_{j=1}^{g}\left[x_j,y_j^{-1}\right]\right)\cdot\widehat{z_i}\right)\right]. \tag{2.8}$$

Pour tout $i \in \{1,\ldots,g\}$, par calcul différentiel libre, nous obtenons :

$$\mathbf{Z}[H] \ni a\frac{\partial}{\partial x_i}\left(\prod_{j=1}^{g}\left[x_j,y_j^{-1}\right]\right) = a\left(\frac{\partial\left[x_i,y_i^{-1}\right]}{\partial x_i}\right) = 1 - y_i^{-1}, \tag{2.9}$$

ainsi que :

$$\mathbf{Z}[H] \ni a\frac{\partial}{\partial y_i}\left(\prod_{j=1}^{g}\left[x_j,y_j^{-1}\right]\right) = a\left(\frac{\partial\left[x_i,y_i^{-1}\right]}{\partial y_i}\right) = y_i^{-1} - x_i y_i^{-1}. \tag{2.10}$$

L'équation (2.7) se déduit de (2.8) et des formules (2.9) et (2.10). □

2.2.3. *Représentation de Magnus du groupe de Torelli de* $\Sigma_{g,1}$. Conservons les notations relatives à $\Sigma_{g,1}$ employées en §2.2.1. Nous notons aussi $\mathcal{T}_{g,1}$ le groupe de Torelli de la surface $\Sigma_{g,1}$. L'assertion c) du Lemme 2.2 suscite notre intérêt pour ce qui suit.

Definition 2.4. La *représentation de Magnus du groupe de Torelli* $\mathcal{T}_{g,1}$ est l'homomorphisme de groupes :

$$\mathcal{T}_{g,1} \xrightarrow{M} GL_{2g}\left(\mathbf{Z}[H]\right),$$

défini par :

$$M(h) = aJ(h_*) = \left(a\left(\frac{\partial h_*(z_j)}{\partial z_i}\right)\right)_{i,j}.$$

Le fait que M soit effectivement un homomorphisme découle de la “règle de la chaîne” du calcul différentiel libre (*cf.* [**Bi**, Theorem 3.9]).
On définit un sous-groupe distingué de $\mathcal{T}_{g,1}$ en posant :

$$\mathcal{D}_{g,1} = \left\{ h \in \mathcal{T}_{g,1} : \frac{\pi}{\pi''} \xrightarrow{h_*} \frac{\pi}{\pi''} \text{ est l'identité} \right\}.$$

THÉORÈME 2.5 (Magnus). *Le noyau de la représentation de Magnus de $\mathcal{T}_{g,1}$ coïncide avec le sous-groupe $\mathcal{D}_{g,1}$.*

DÉMONSTRATION. Le fait suivant, bien connu, remonte à Magnus (*cf.* [**Mg**]) : il est exprimé ici avec le formalisme du calcul différentiel libre et nous en donnons une preuve explicitement topologique.

AFFIRMATION 2.6. Soient $x, y \in \pi$, alors $x \equiv y \bmod \pi''$ si et seulement si :

$$\forall i \in \{1, \ldots, n\},\ a\left(\frac{\partial x}{\partial z_i}\right) = a\left(\frac{\partial y}{\partial z_i}\right) \in \mathbf{Z}[\pi/\pi'].$$

DÉMONSTRATION DE L'AFF. 2.6. Pour tous $x, y \in \pi$ et pour tout $i \in \{1, \ldots, n\}$, nous avons :

$$a\left(\frac{\partial x y^{-1}}{\partial z_i}\right) = a\left(\frac{\partial x}{\partial z_i}\right) - a(x)a(y^{-1})a\left(\frac{\partial y}{\partial z_i}\right);$$

de plus, on déduit de la relation fondamentale du calcul différentiel libre que :

$$\begin{cases} a(x) - 1 = \sum_{i=1}^n a\left(\frac{\partial x}{\partial z_i}\right) \cdot (z_i - 1) \in \mathbf{Z}[\pi/\pi'], \\ a(y) - 1 = \sum_{i=1}^n a\left(\frac{\partial y}{\partial z_i}\right) \cdot (z_i - 1) \in \mathbf{Z}[\pi/\pi']. \end{cases}$$

Il suffit donc de démontrer que pour tout $x \in \pi$:

$$(x \in \pi'') \iff \left(\forall i \in \{1, \ldots, n\},\ a\left(\frac{\partial x}{\partial z_i}\right) = 0 \in \mathbf{Z}[\pi/\pi']\right). \tag{2.11}$$

Soit pour cela X le bouquet de n cercles $z_1, \ldots, z_n$ basés en $\star$ (comme dans la preuve du Lemme 2.2) et soit $\left(\widehat{X}, p\right)$ le revêtement abélien maximal de $(X, \star)$. Supposons que $x \in \pi \simeq \pi_1(X, \star)$ est représenté par un lacet $\gamma \subset X$ basé en $\star$ et soit $\widehat{\gamma}$ son unique relèvement débutant en $\widehat{\star}$. Alors, $x \in \pi''$ si et seulement si $\widehat{\gamma}$ est un lacet dans $\widehat{X}$ et si ce dernier est nul-homologue dans $\widehat{X}$. Puisque $\widehat{X}$ est 1-dimensionnel, cette dernière condition équivaut à demander : $\widehat{\gamma} = 0 \in C_1\left(\widehat{X}\right)$. On déduit alors l'équivalence (2.11) de la formule (2.6) rappelée dans la preuve du Lemme 2.2. □

Pour tout $h \in \mathcal{T}_{g,1}$, $h \in \mathcal{D}_{g,1}$ si et seulement si $h_*(z_i) \equiv z_i$ modulo π'' pour tout $i \in \{1, \ldots, n\}$. Le théorème de Magnus est donc une conséquence de l'Aff. 2.6. □

2.3. Calcul de variation. Nous sommes maintenant prêts à résoudre le problème posé en introduction de cette section (page 155). Rappelons-nous qu'il s'agissait d'évaluer la variation entre les torsions $\tau^{\varphi}(M, \alpha)$ et $\tau^{\varphi'}(M', \alpha')$ grâce à la formule contingente de Mayer-Vietoris.

Les résultats du §2.2 nous invite d'ores et déjà à distinguer le cas “exceptionnel” où l'homomorphisme de groupes multiplicatifs :

$$H_1(\partial M_2) \longrightarrow H_1(M) \xrightarrow{\varphi} \mathbf{F}^*$$

est trivial, du cas “générique” où cet homomorphisme de groupes est non-trivial.

2.3.1. *Calcul de la variation dans le cas "générique".* Nous commençons par donner, une condition nécessaire et suffisante à la nullité de $\tau^{\varphi}(M, \alpha)$ dans le cas "générique". Pour cela, nous notons les inclusions comme suit :

$$\begin{tikzcd}
 & M_1 \arrow[dr, "i_1"] & \\
\partial M_1 \arrow[ur, "j_1"] \arrow[rr, "l_1"] & & M \arrow[d, equal] \\
\partial M_2 \arrow[u, "f", "\simeq"'] \arrow[rr, "l_2"] \arrow[dr, "j_2"'] & & M \\
 & M_2 \arrow[ur, "i_2"'] &
\end{tikzcd}$$

Nous fixons dans la suite un point $\star$ sur ∂M_2 (les points $f(\star)$, $j_2(\star)$, $j_1 f(\star)$ et $l_2(\star)$ sont encore notés $\star$). Pour $k = 1$ ou 2, nous posons aussi :

$$I_k(\star) = Ker\left(H_1^{\varphi l_{k,*}}(\partial M_k, \star) \xrightarrow{j_{k,*}} H_1^{\varphi i_{k,*}}(M_k, \star)\right).$$

LEMME 2.7. *Dans le cas "générique", $H_*^{\varphi}(M, \star) = 0$ si et seulement si les deux conditions suivantes sont satisfaites :*

– $\forall r \neq 1,\ \forall k \in \{1, 2\},\quad H_r^{\varphi i_{k,*}}(M_k, \star) = 0,$

– $f_*^{-1}(I_1(\star)) \cap I_2(\star) = 0,$

auquel cas nous avons : $H_1^{\varphi l_{2,*}}(\partial M_2, \star) = f_*^{-1}(I_1(\star)) \oplus I_2(\star)$.

DÉMONSTRATION. D'après le Lemme 2.1, $H_*^{\varphi}(M, \star) = 0$ si et seulement si :

$$H_r^{\varphi l_{2,*}}(\partial M_2, \star) \xrightarrow{(j_{1,*} \circ f_*, j_{2,*})} H_r^{\varphi i_{1,*}}(M_1, \star) \oplus H_r^{\varphi i_{2,*}}(M_2, \star)$$

est un isomorphisme pour tout r. Etant dans le cas "générique", nous avons d'après le Lemme 2.3 a) :

$$H_r^{\varphi l_{2,*}}(\partial M_2, \star) = 0$$

pour $r \neq 1$, et $\dim\left(H_1^{\varphi l_{2,*}}(\partial M_2, \star)\right) = 2g - 2$, si g est le genre de la surface ∂M_2. De plus, $\chi(M_k) = 1 - g$. L'équivalence annoncée découle donc du fait, qu'en dimension $r = 1$, le noyau de $(j_{1,*} \circ f_*, j_{2,*})$ est $f_*^{-1}(I_1(\star)) \cap I_2(\star)$.

Supposons maintenant qu'on ait $H_*^{\varphi}(M, \star) = 0$, alors :

$$\dim(I_k(\star)) = \dim\left(H_1^{\varphi l_{k,*}}(\partial M_k, \star)\right) - \dim\left(H_1^{\varphi i_{k,*}}(M_k, \star)\right) = (2g-2) - (g-1) = g - 1.$$

Donc, $\dim(I_2(\star)) + \dim\left(f_*^{-1}(I_1(\star))\right) = \dim\left(H_1^{\varphi l_{2,*}}(\partial M_2, \star)\right)$, si bien que :

$$H_1^{\varphi l_{2,*}}(\partial M_2, \star) = f_*^{-1}(I_1(\star)) \oplus I_2(\star).$$

$\square$

Lorsque $\tau^{\varphi}(M, \alpha) \neq 0$, nous avons par convention :

$$H_*^{\varphi}(M, \star) = 0$$

et donc d'après le Lemme 2.7 :

$$H_1^{\varphi l_{2,*}}(\partial M_2, \star) = f_*^{-1}(I_1(\star)) \oplus I_2(\star).$$

Soit alors $H_1^{\varphi l_{2,*}}(\partial M_2,\star) \xrightarrow{pr_2} I_2(\star)$ la projection sur le deuxième facteur de cette somme directe. Nous pouvons considérer l'endomorphisme $pr_2 \circ h_*|_{I_2(\star)}$ du $\mathbf{F}$-espace vectoriel $I_2(\star)$ puis son déterminant :

$$\det\left(pr_2 \circ h_*|_{I_2(\star)}\right) \in \mathbf{F}.$$

THÉORÈME 2.8. *Dans le cas "générique", supposons que :*

$$\tau^{\varphi}(M,\alpha) \neq 0.$$

Il existe alors une constante $C \in \mathbf{F}$ indépendante de α et égale à $\det\left(pr_2 \circ h_*|_{I_2(\star)}\right)$, à un facteur multiplicatif près dans $\varphi(H_1(M))$, telle que :

$$\tau^{\varphi'}(M',\alpha') = C \cdot \tau^{\varphi}(M,\alpha) \in \mathbf{F}.$$

DÉMONSTRATION. Dans cette preuve, afin de simplifier la notation des groupes d'homologie à coefficients twistés, nous n'indiquons pas la dépendance de ceux-ci en le point $\star$.

Pour tout $x \in H^2(M)$, nous avons :

$$\begin{aligned} \frac{\tau^{\varphi'}(M',\Omega(x\cdot\alpha))}{\tau^{\varphi}(M,x\cdot\alpha)} &= \frac{\tau^{\varphi'}(M',P\Phi P^{-1}(x)\cdot\Omega(\alpha))}{\tau^{\varphi}(M,x\cdot\alpha)} \\ &= \frac{\varphi'(\Phi P^{-1}(x))\cdot\tau^{\varphi'}(M',\Omega(\alpha))}{\varphi(P^{-1}(x))\cdot\tau^{\varphi}(M,\alpha)} \\ &= \frac{\tau^{\varphi'}(M',\Omega(\alpha))}{\tau^{\varphi}(M,\alpha)}, \end{aligned}$$

ce qui démontre la première assertion.

Soient, pour $i=1$ et 2, une C^1-triangulation (K_i,ρ_i) de la variété lisse M_i, telles que :

– (K_i,ρ_i) est une extension d'une triangulation (L_i,ρ_i) de ∂M_i,

– $L_2 \xrightarrow{(\rho_1)^{-1} f \rho_2} L_1$ est un isomorphisme simplicial.

Soit la C^1-triangulation de M :

$$(K,\rho)$$

obtenue en recollant les triangulations (K_1,ρ_1) et (K_2,ρ_2) par f.

Soit une orientation homologique ω de $|K|$ telle que $\rho_*(\omega)=\omega_M$. Piochons aussi e une structure d'Euler combinatoire sur K. Alors :

$$\tau^{\varphi}(M,\alpha) \equiv \tau^{\varphi\rho_*}(K;e,\omega) \in \frac{\mathbf{F}^*}{\varphi\left(H_1(M)\right)}.$$

Nous appliquons maintenant la formule contingente de Mayer-Vietoris en maintenant les notations de §2.1 :

$$\tau^{\varphi\rho_*}(K;e,\omega) = \epsilon \cdot \frac{\check{\tau}^{\varphi i_{1,*}\rho_{1,*}}\left(K_1,\star;\mathfrak{e}_1,b_1,\omega_1\right)\cdot\check{\tau}^{\varphi i_{2,*}\rho_{2,*}}\left(K_2,\star;\mathfrak{e}_2,b_2,\omega_2\right)}{\check{\tau}^{\varphi l_{2,*}\rho_{2,*}}\left(L_2,\star;\mathfrak{f}_2,c_2,\varpi_2\right)}, \tag{2.12}$$

où le sous-complexe "$K_1 \cap K_2$" de K a été identifié avec L_2.

Intéressons-nous maintenant à M'. Quitte à modifier h dans sa classe d'isotopie, nous pouvons supposer que $\star$ est fixé par le difféomorphisme h. Nous notons comme suit les inclusions :

$$
\begin{tikzcd}
 & M_1 \arrow[dr, "i'_1"] & \\
\partial M_1 \arrow[ur, "j'_1"] \arrow[rr, "l'_1"] & & M' \arrow[d, equal] \\
\partial M_2 \arrow[u, "f \circ h", "\simeq"'] \arrow[rr, "l'_2"] \arrow[dr, "j'_2"'] & & M' \\
 & M_2 \arrow[ur, "i'_2"'] &
\end{tikzcd}
$$

Le point $l'_2(\star) \in M'$ est encore noté $\star$. Nous avons bien sûr $j'_1 = j_1$ et $j'_2 = j_2$. Remarquons aussi que par définition de l'isomorphisme d'homologie Φ : $\varphi' i'_k = \varphi i_k$ et $\varphi' l'_k = \varphi l_k$. Ayant supposé $H_*^{\varphi}(M) = 0$, on déduit d'une double application du Lemme 2.7 que :

$$\left(\tau^{\varphi'}(M', \alpha') \neq 0\right) \iff \left(h_*^{-1} f_*^{-1}(I_1) \cap I_2 = 0\right).$$

Alors :

$$
\begin{aligned}
\left(\tau^{\varphi'}(M', \alpha') \neq 0\right) &\iff \left(f_*^{-1}(I_1) \cap h_*(I_2) = 0\right) \\
&\iff (\forall y \in h_*(I_2),\ pr_2(y) = 0 \Rightarrow y = 0) \\
&\iff (\forall x \in I_2,\ pr_2\,(h_*(x)) = 0 \Rightarrow x = 0) \\
&\iff (pr_2 \circ h_*|_{I_2} \text{ est injective }) \\
&\iff (\det(pr_2 \circ h_*|_{I_2}) \neq 0).
\end{aligned}
$$

Ces équivalences prouvent le théorème dans le cas où $\tau^{\varphi'}(M', \alpha') = 0$. Nous supposerons donc dorénavant que $\tau^{\varphi'}(M', \alpha') \neq 0$.

Nous allons construire à partir de la triangulation (K, ρ) de M une triangulation (K', ρ') de M'.

Affirmation 2.9. Il existe, quitte à modifier h dans sa classe d'isotopie, des subdivisions L'_1 de L_1 et L'_2 de L_2 telles que la composée $L'_2 \xrightarrow{(\rho_1)^{-1} f h \rho_2} L'_1$ soit un isomorphisme simplicial.

Démonstration de l'Aff. 2.9. Nous disposons de deux C^1-triangulations de ∂M_1 : (L_1, ρ_1) d'une part et $(L_2, fh\rho_2)$ d'autre part. D'après [**Mu**, §10.5 et §10.9], il existe des C^1-triangulations (L'_1, ρ'_1) et (L'_2, ρ'_2) de ∂M_1 telles que :

– L'_i est une subdivision de L_i pour $i = 1, 2$;

– ρ'_1 est homotope à ρ_1, et ρ'_2 à $fh\rho_2$;

– la composée $L'_2 \xrightarrow{(\rho'_1)^{-1} \circ \rho'_2} L'_1$ est un isomorphisme simplicial.

Il existe donc un difféomorphisme $\partial M_1 \xrightarrow[\simeq]{g} \partial M_1$, homotope à l'identité telle que $(\rho_1)^{-1} \circ g \circ (fh\rho_2) : L'_2 \longrightarrow L'_1$ soit un isomorphisme simplicial. Or :

$$(\rho_1)^{-1} g f h \rho_2 \quad = \quad (\rho_1)^{-1} f (f^{-1} g f h) \rho_2,$$

et le difféomorphisme $f^{-1} g f h = (f^{-1} g f) h$ est homotope à h, et est donc isotope à h. □

Soit K'_i une subdivision de K_i étendant la subdivision L'_i de L_i pour $i = 1$ et 2. Nous construisons alors :

$$(K', \rho'),$$

triangulation de la variété M', en recollant (K'_1, ρ_1) avec (K'_2, ρ_2) par fh.

Soit une orientation homologique ω' de $|K'|$ correspondant à $\omega_{M'}$ par ρ'_*. Piochons aussi une structure d'Euler combinatoire e' sur K'. Alors :

$$\tau^{\varphi'}(M', \alpha') \equiv \tau^{\varphi' \rho'_*}(K'; e', \omega') \in \frac{\mathbf{F}^*}{\varphi\left(H_1(M)\right)}.$$

Et par application de la formule contingente de Mayer-Vietoris, en primant les notations de §2.1, on obtient :

(2.13)

$$\tau^{\varphi' \rho'_*}(K'; e', \omega') = \epsilon' \cdot \frac{\check{\tau}^{\varphi i_{1,*} \rho_{1,*}}\left(K'_1, \star; \mathfrak{e}'_1, b'_1, \omega'_1\right) \cdot \check{\tau}^{\varphi i_{2,*} \rho_{2,*}}\left(K'_2, \star; \mathfrak{e}'_2, b'_2, \omega'_2\right)}{\check{\tau}^{\varphi l_{2,*} \rho_{2,*}}\left(L'_2, \star; \mathfrak{f}'_2, c'_2, \varpi'_2\right)},$$

Ici, le sous-complexe "$K'_1 \cap K'_2$" de K' est identifié avec L'_2, et on a utilisé les faits suivants : $\varphi i_{k,*} = \varphi' i'_{k,*}$, $\varphi l_{2,*} = \varphi' l'_{2,*}$.

Nous comparons maintenant les formules (2.12) et (2.13), à une facteur multiplicatif près dans $\varphi\left(H_1(M)\right)$.

Tout d'abord, d'après le Lemme 2.1, les signes ϵ et ϵ' sont égaux car ils dépendent pareillement des mêmes classes d'isomorphismes d'espaces vectoriels gradués.

Ensuite, on peut choisir $\omega'_1 = \omega_1$, $\omega'_2 = \omega_2$ et $\varpi'_2 = \varpi_2$. En effet, soit $\mathcal{H}$ (respectivement $\mathcal{H}'$) la suite longue de Mayer-Vietoris associée à la réunion $M = M_1 \cup_f M_2$ (respectivement à $M' = M_1 \cup_{fh} M_2$) pour les coefficients réels. On définit un isomorphisme η entre les $\mathbf{R}$-complexes acycliques $\mathcal{H}$ et $\mathcal{H}'$ en posant dimension par dimension de ces complexes :

$$\Big(H_r(\partial M_2; \mathbf{R}) \xrightarrow{\ \eta\ } H_r(\partial M_2; \mathbf{R})\Big) := Id, \quad \text{pour tout } r,$$

$$\Big(H_r(M_1; \mathbf{R}) \oplus H_r(M_2; \mathbf{R}) \xrightarrow{\ \eta\ } H_r(M_1; \mathbf{R}) \oplus H_r(M_2; \mathbf{R})\Big) := Id, \quad \text{pour tout } r,$$

$$\Big(H_r(M; \mathbf{R}) \xrightarrow{\ \eta\ } H_r(M'; \mathbf{R})\Big) := \begin{cases} ([pt] \longmapsto [pt]) & \text{pour } r = 0, \\ \Phi & \text{pour } r = 1, \\ \left(\Phi^\sharp\right)^{-1} & \text{pour } r = 2, \\ ([M] \longmapsto [M']) & \text{pour } r = 3. \end{cases}$$

Ici, $\Phi^\sharp$ est l'isomorphisme dual de Φ pour les formes d'intersection de M et de M'. On conclut grâce à la définition des orientations homologiques ω_M et $\omega_{M'}$.

Ayant supposé $H_*^{\varphi'}(M') = 0$, les bases homologiques b'_1, b'_2 et c'_2 sont, d'après le Lemme 2.7, concentrées en degré 1. Lors du *bon choix*, nous avons pu demander $b'_1 = b_1$ et $b'_2 = b_2$ (c'_2 nous est alors imposée).

Nous déduisons alors des équations (2.12), (2.13) et de l'invariance par subdivision énoncée par le Lemme 1.18 que :

$$\text{(2.14)} \qquad \frac{\tau^{\varphi' \rho'_*}(K'; e', \omega')}{\tau^{\varphi \rho_*}(K; e, \omega)} \equiv [c_2/c'_2] \in \frac{\mathbf{F}^*}{\varphi\left(H_1(M)\right)}.$$

Affirmation 2.10.

$$[c_2/c'_2] = \det\left(pr_2 \circ h_*|_{I_2}\right) \in \mathbf{F}^*.$$

Démonstration de l'Aff. 2.10. Par le choix des bases c_2 et c'_2 de l'espace vectoriel $H_1^{\varphi l_{2,*}}(\partial M_2)$, $[c_2/c'_2]$ est égal au déterminant de la composée suivante :

$$H_1^{\varphi l_{2,*}}(\partial M_2) \xrightarrow[\simeq]{(j_{1,*} f_* h_*, j_{2,*})} H_1^{\varphi i_{1,*}}(M_1) \oplus H_1^{\varphi i_{2,*}}(M_2) \xrightarrow[\simeq]{(j_{1,*} f_*, j_{2,*})^{-1}} H_1^{\varphi l_{2,*}}(\partial M_2).$$

Soit d_1 une base de I_1 et d_2 une base de I_2 si bien que $(f_*^{-1}(d_1), d_2)$ est une base de $H_1^{\varphi l_{2,*}}(\partial M_2)$. Notons :

$$H = \left(\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right)$$

la matrice de h_* dans cette base. En particulier, H_{22} est la matrice de $pr_2 \circ h_*|_{I_2}$ dans la base d_2.
La matrice de $(j_{1,*}f_*, j_{2,*})$ dans les bases $(f_*^{-1}(d_1), d_2)$ et $(j_{1,*}f_*(d_2), j_{2,*}f_*^{-1}(d_1))$ est :

$$\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)$$

et celle de $(j_{1,*}f_*h_*, j_{2,*})$ dans les bases $(f_*^{-1}(d_1), d_2)$ et $(j_{1,*}f_*(d_2), j_{2,*}f_*^{-1}(d_1))$ est :

$$\left(\begin{array}{c|c} H_{21} & H_{22} \\ \hline 1 & 0 \end{array}\right)$$

On en déduit donc que :

$$\begin{aligned} [c_2/c_2'] &= \det\left(\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)^{-1} \cdot \left(\begin{array}{c|c} H_{21} & H_{22} \\ \hline 1 & 0 \end{array}\right)\right) \\ &= \det\left(\begin{array}{c|c} 1 & 0 \\ \hline H_{21} & H_{22} \end{array}\right) \\ &= \det(H_{22}). \end{aligned}$$

□

L'Affirmation 2.10 et l'équation (2.14) achèvent la preuve du théorème. □

Nous généralisons maintenant quelque peu le Th. 2.8.

THÉORÈME 2.11. *Dans le cas "générique", supposons qu'il existe des applications* $\mathbf{F}$*-linéaires :*

$$I_2(\star) \xrightarrow{r_1} f_*^{-1}(I_1(\star)) \quad \textit{et} \quad I_2(\star) \xrightarrow{r_2} I_2(\star),$$

telles que :

$$\forall x \in I_2(\star), \quad h_*(x) = r_1(x) + r_2(x) \in H_1^{\varphi l_{2,*}}(\partial M_2, \star).$$

Il existe alors une constante $C \in \mathbf{F}$ *indépendante de* α *et égale à* $\det(r_2)$*, à un facteur multiplicatif près dans* $\varphi(H_1(M))$*, telle que :*

$$\tau^{\varphi'}(M', \alpha') = C \cdot \tau^{\varphi}(M, \alpha) \in \mathbf{F}.$$

DÉMONSTRATION. Lorsque $\tau^{\varphi}(M,\alpha) \neq 0$, le théorème est une reformulation du Th. 2.8 ; il nous suffit donc de montrer que : $(\tau^{\varphi}(M,\alpha) = 0) \Rightarrow \left(\tau^{\varphi'}(M',\alpha') = 0\right)$. Nous allons en fait démontrer que :

$$\left(\tau^{\varphi'}(M', \alpha') \neq 0\right) \Longrightarrow \left(\tau^{\varphi}(M, \alpha) \neq 0 \text{ et } \det(r_2) \neq 0\right).$$

D'après une double application du Lemme 2.7, cela découlera de l'assertion suivante :

$$\left(f_*^{-1}(I_1(\star)) \cap h_*(I_2(\star)) = 0\right) \Longrightarrow \left(f_*^{-1}(I_1(\star)) \cap I_2(\star) = 0 \text{ et } \det(r_2) \neq 0\right).$$

Supposons donc que $f_*^{-1}(I_1(\star)) \cap h_*(I_2(\star)) = 0$. Soit $x \in I_2(\star)$ tel que $r_2(x) = 0$; alors :

$$h_*(I_2(\star)) \ni h_*(x) = r_1(x) \in f_*^{-1}(I_1(\star)),$$

donc $h_*(x) = 0$ et $x = 0$. Ceci prouve l'injectivité, et donc l'inversibilité de r_2. Soit maintenant $x \in f_*^{-1}(I_1(\star)) \cap I_2(\star)$; alors :

$$h_*(I_2(\star)) \ni h_*\left(r_2^{-1}(x)\right) = \underbrace{x}_{\in f_*^{-1}(I_1(\star))} + \underbrace{r_1\left(r_2^{-1}(x)\right)}_{\in f_*^{-1}(I_1(\star))} \in f_*^{-1}(I_1(\star)).$$

On en déduit que $h_*\left(r_2^{-1}(x)\right) = 0$, puis que $r_2^{-1}(x) = 0$ et enfin que $x = 0$. Aussi avons-nous bien : $f_*^{-1}(I_1(\star)) \cap I_2(\star) = 0$. $\square$

2.3.2. *Calcul de la variation dans le cas "exceptionnel".* Nous envisageons maintenant le cas "exceptionnel", plus facile que le cas "générique".

THÉORÈME 2.12. *Dans le cas "exceptionnel", il existe une constante $C \in \mathbf{F}$ indépendante de α et appartenant à $\varphi(H_1(M))$, telle que :*

$$\tau^{\varphi'}(M', \alpha') = C \cdot \tau^{\varphi}(M, \alpha) \in \mathbf{F}.$$

DÉMONSTRATION. D'après le Lemme 2.1, $\tau^{\varphi}(M, \alpha)$ est non-nulle si et seulement si :

$$H_*^{\varphi l_{2,*}}(\partial M_2, \star) \xrightarrow{(j_{1,*} \circ f_*, j_{2,*})} H_*^{\varphi i_{1,*}}(M_1, \star) \oplus H_*^{\varphi i_{2,*}}(M_2, \star)$$

est un isomorphisme, et $\tau^{\varphi'}(M', \alpha')$ est non-nulle si et seulement si :

$$H_*^{\varphi l_{2,*}}(\partial M_2, \star) \xrightarrow{(j_{1,*} \circ f_* h_*, j_{2,*})} H_*^{\varphi i_{1,*}}(M_1, \star) \oplus H_*^{\varphi i_{2,*}}(M_2, \star)$$

est un isomorphisme. Or dans notre cas, $H_*^{\varphi l_{2,*}}(\partial M_2, \star) = H_*(\partial M_2; \mathbf{F})$ si bien que :

$$H_*^{\varphi l_{2,*}}(\partial M_2, \star) \xrightarrow{h_*} H_*^{\varphi l_{2,*}}(\partial M_2, \star)$$

est l'identité. Les torsions $\tau^{\varphi}(M, \alpha)$ et $\tau^{\varphi'}(M', \alpha')$ s'annulent donc en même temps. On les suppose dans la suite non-nulles.

En répétant les arguments de la preuve du Th. 2.8, qui se simplifient ici vu que h_* est l'identité, nous arrivons à :

$$\frac{\tau^{\varphi'}(M', \alpha')}{\tau^{\varphi}(M, \alpha)} \in \varphi(H_1(M)).$$

On montre de même qu'au début de la preuve du Th. 2.8, que ce rapport est indépendant de α. $\square$

2.3.3. *Un cas particulier.* Soient g le genre de la surface ∂M_2 et une inclusion $\Sigma_{g,1} \xhookrightarrow{j} \partial M_2$. Quitte à modifier $h \in \mathcal{T}(\partial M_2)$ dans sa classe d'isotopie, on peut supposer qu'il existe $k \in \mathcal{T}_{g,1}$ tel que :

$$\forall x \in \partial M_2,\ h(x) = \begin{cases} jkj^{-1}(x) & \text{si } x \in j(\Sigma_{g,1}), \\ x & \text{sinon.} \end{cases}$$

COROLLAIRE 2.13. *Supposons que k appartient au sous-groupe $\mathcal{D}_{g,1}$ de $\mathcal{T}_{g,1}$ défini au §2.2.3. Il existe alors une constante $C \in \mathbf{F}$ indépendante de α et appartenant à $\varphi(H_1(M))$, telle que :*

$$\tau^{\varphi'}(M', \alpha') = C \cdot \tau^{\varphi}(M, \alpha) \in \mathbf{F}.$$

DÉMONSTRATION. Dans le cas "exceptionnel" où la composition :

$$H_1(\partial M_2) \xrightarrow{l_{2,*}} H_1(M) \xrightarrow{\varphi} \mathbf{F}^*$$

est triviale, il suffit d'appliquer le Th. 2.12.

Plaçons-nous dans le cas "générique". Puisque $M(h)$ est la matrice identité (Th. 2.5), il découle alors du Lemme 2.3 c) que :

$$H_1^{\varphi l_{2,*}}(\partial M_2, \star) \xrightarrow{h_*} H_1^{\varphi l_{2,*}}(\partial M_2, \star)$$

est l'identité. On conclut alors grâce au Th. 2.11. □

2.3.4. *Indétermination sur la variation de la torsion.* Nous concluons sur une remarque générale concernant nos précédents résultats. Chacun d'entre-eux s'attachait à trouver $C \in \mathbf{F}$ telle que :

$$\tau^{\varphi'}(M', \alpha') = C \cdot \tau^{\varphi}(M, \alpha) \in \mathbf{F}.$$

La constante C, indépendante de α, n'était reconnue qu'à un facteur multiplicatif près dans $\varphi(H_1(M))$. Soit :

$$Q \in \frac{\mathbf{F}}{\varphi(H_1(M))}$$

cette classe d'équivalence. Nous rendons la discussion intéressante en supposant dans la suite que $\tau^{\varphi}(M, \alpha) \neq 0$. Le lemme suivant montre que, sous certaines hypothèses sur $\mathbf{F}$ et φ, *cette indétermination peut être en fait levée.*

LEMME 2.14. *Supposons les hypothèses suivantes satisfaites :*

(1) *le corps* $\mathbf{F}$ *est muni d'une involution* $\left(f \longmapsto \overline{f}\right)$ *telle que :*

$$\forall h \in H_1(M), \quad \varphi\left(h^{-1}\right) = \overline{\varphi(h)};$$

(2) $-1 \notin \varphi(H_1(M))$.

Alors, la constante C *est l'unique représentant de la classe* Q *qui soit* symétrique, *c'est-à-dire qui vérifie :*

$$\overline{C} = C \in \mathbf{F}.$$

EXEMPLE 2.15. Ainsi, sous ces hypothèses sur $\mathbf{F}$ et φ, les constantes C apparaissant dans le Th. 2.12 et dans le Cor. 2.13 sont égales à 1.
Ces hypothèses sont par exemple satisfaites pour la torsion de Milnor (introduite dans l'Ex. 1.24).

DÉMONSTRATION DU LEMME 2.14. La constante C est effectivement symétrique, car :

$$\begin{aligned}
\overline{C} &= \overline{\tau^{\varphi'}(M', \alpha')}/\overline{\tau^{\varphi}(M, \alpha)} \\
&= \left(\varphi'\left(P^{-1}c(\alpha')\right)^{-1} \cdot \tau^{\varphi'}(M', \alpha')\right) / \left(\varphi\left(P^{-1}c(\alpha)\right)^{-1} \cdot \tau^{\varphi}(M, \alpha)\right) \\
&= \tau^{\varphi'}(M', \alpha')/\tau^{\varphi}(M, \alpha) \\
&= C.
\end{aligned}$$

Ici, on a fait une double application du Théorème 1.25 et on a utilisé le fait que $c(\alpha') = P\Phi P^{-1}c(\alpha)$. Ensuite, pour tout $h \in H_1(M)$, nous avons :

$$\overline{\varphi(h) \cdot C} = \varphi(h)^{-1} \cdot \overline{C} = \varphi(h)^{-2} \cdot (\varphi(h) \cdot C).$$

Donc, le représentant $\varphi(h) \cdot C$ de Q est symétrique si et seulement si $\varphi(h)^2 = 1$. Par hypothèse, C est donc l'unique représentant de Q qui soit symétrique. □

3. Torsions abéliennes et clovers bouclés

DEFINITION 3.1. Un clover G dans une 3-variété compacte orientée M est dit *bouclé* lorsque son degré de boucles est non-nul :

$$l\text{-}deg(G) \geq 1.$$

Les chirurgies sur les clovers bouclés engendrent parmi les 3-variétés une relation d'équivalence bien plus fine que la relation de Y-équivalence, comme l'enseigne le théorème suivant.

THÉORÈME 3.2. (Levine, [**Le**, Th. 2]) *Soit* G *un clover bouclé dans une* 3-*variété compacte orientée* M. *Alors,* M *et* M_G *sont homologiquement cobordantes,* i.e. *il existe une* 4-*variété compacte orientée* V *de bord* $\partial V = M \dot\cup (-M_G)$, *telle que les inclusions* $M \hookrightarrow V$ *et* $M_G \hookrightarrow V$ *sont des équivalences d'homologie.*

Grâce aux résultats de la section précédente, nous évaluons ici la variation subie par les torsions abéliennes de Reidemeister-Turaev d'une $Spin^c$-variété fermée, lors d'une chirurgie le long d'un clover bouclé. Puis, on en déduit que ces torsions sont *multiplicativement* des invariants de degré 1 pour ce type de mouvement chirurgical. Pour cela, nous calculons au préalable l'image du difféomorphisme Borroméen par la représentation de Magnus.

3.1. Calculs autour du difféomorphisme Borroméen. Nous rappelons que le difféomorphisme Borroméen, tel qu'il a été défini dans la Remarque 2.8 du Chapitre 2, est un difféomorphisme positif de Σ_3. Nous le considérons maintenant comme un difféomorphisme positif de $\Sigma_{3,1}$:

$$\Sigma_{3,1} \xrightarrow[\simeq]{h} \Sigma_{3,1}$$

en posant :

$$h := h_b^{-1} \circ h_a \quad \text{avec} \quad h_a = \prod_{i=1}^{3} \tau_{\alpha_i} \quad \text{et} \quad h_b = \prod_{i=1}^{3} \tau_{\beta_i},$$

où α_i et β_i sont les courbes de $\Sigma_{3,1}$ dessinées sur la Figure 3.1.

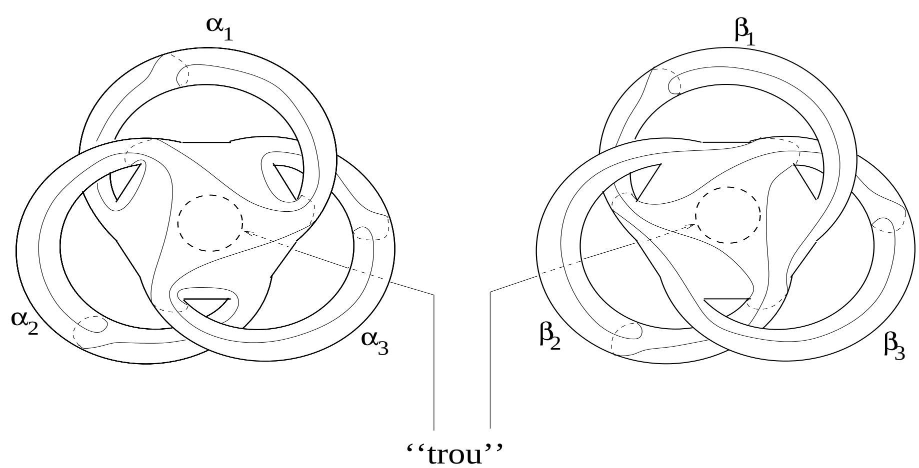

FIG. 3.1 – Les courbes α_i et β_i

3.1.1. *Calcul du difféomorphisme Borroméen au niveau du* π_1. Soit $\star' \in \Sigma_{3,1}$ et $(x_i', y_i')_{i=1}^3$ les lacets basés en $\star'$ tels que tracés sur la Figure 3.2. Alors, $\pi_1(\Sigma_{3,1}, \star')$ est un groupe libre sur la famille $(x_i', y_i')_{i=1}^3$.
Le point base $\star'$, tel qu'il a été choisi en dehors de $\partial\Sigma_{3,1}$, continue à faire jouer des rôles symétriques aux anses 1, 2 et 3. Pour les calculs à venir, nous aurons besoin de choisir un point base sur le bord de $\Sigma_{3,1}$. Soient alors le point $\star \in \partial\Sigma_{3,1}$ et le chemin orienté γ reliant $\star'$ à $\star$ qui sont dessinés sur la Figure 3.3. Nous définissons

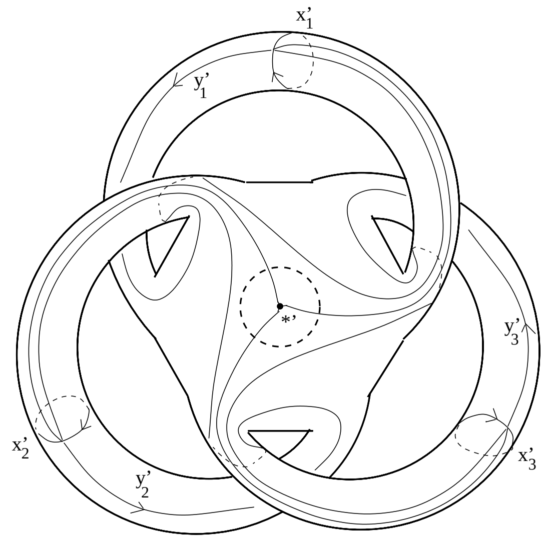

FIG. 3.2 – La base $(x'_i, y'_i)_{i=1}^3$ de $\pi_1(\Sigma_{3,1}, \star')$

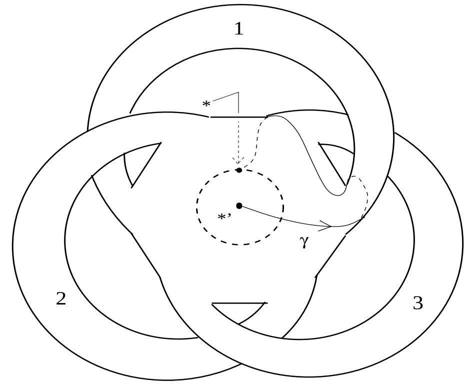

FIG. 3.3 – Changement de point base qui brise la symétrie

aussi pour tout $i \in \{1, 2, 3\}$ les lacets :

$$\left\{ \begin{array}{l} x_i = c_{\overline{\gamma}}(x'_i) = \overline{\gamma} \cdot x'_i \cdot \gamma \\ y_i = c_{\overline{\gamma}}(y'_i) = \overline{\gamma} \cdot y'_i \cdot \gamma \end{array} \right.$$

qui ont été dessinés sur la Figure 3.4 après une isotopie des anses. Alors, $\pi_1(\Sigma_{3,1}, \star)$ est un groupe libre sur la famille $(x_i, y_i)_{i=1}^3$.

LEMME 3.3. *Soit* H *l'endomorphisme du groupe libre* $\pi_1(\Sigma_{3,1}, \star)$ *défini par :*

$$\left\{ \begin{array}{rcl} \mathsf{H}(x_i) & = & x_i^{-1} y_{i+2}^{-1} x_{i+2}^{-1} y_{i+2} x_i x_{i+1}^{-1} y_{i+1} x_{i+2} x_i^{-1} y_i x_{i+1} x_{i+2}^{-1} y_{i+1}^{-1} x_{i+1}^{-1} y_{i+1} x_{i+2} x_i \\ & & x_{i+2}^{-1} y_{i+1}^{-1} x_{i+1} x_i^{-1} y_{i+2}^{-1} x_{i+2} y_{i+2} x_i x_{i+1}^{-1} y_i^{-1} x_i^{-1} y_i x_{i+1} x_{i+2}^{-1} y_{i+2} x_i \\ & & x_{i+1}^{-1} y_{i+1} x_{i+2} x_i^{-1} y_{i+2}^{-1} x_{i+2}^{-1} y_{i+2} x_i x_{i+1} x_i^{-1} y_{i+2}^{-1} x_{i+2} x_{i+1}^{-1} y_i^{-1} x_i^2 x_{i+2}^{-1} \\ & & y_{i+1}^{-1} x_{i+1} x_i^{-1} y_{i+2}^{-1} x_{i+2} y_{i+2} x_i \\ \mathsf{H}(y_i) & = & x_i^{-1} y_{i+2}^{-1} x_{i+2}^{-1} y_{i+2} x_i x_{i+1}^{-1} y_{i+1} x_{i+2} x_i^{-1} y_i x_{i+1} x_{i+2}^{-1} y_{i+1}^{-1} x_{i+1}^{-1} y_{i+1} x_{i+2} x_i \\ & & x_{i+2}^{-1} y_{i+1}^{-1} x_{i+1} x_i^{-1} y_{i+2}^{-1} x_{i+2} y_{i+2} x_i \end{array} \right.$$

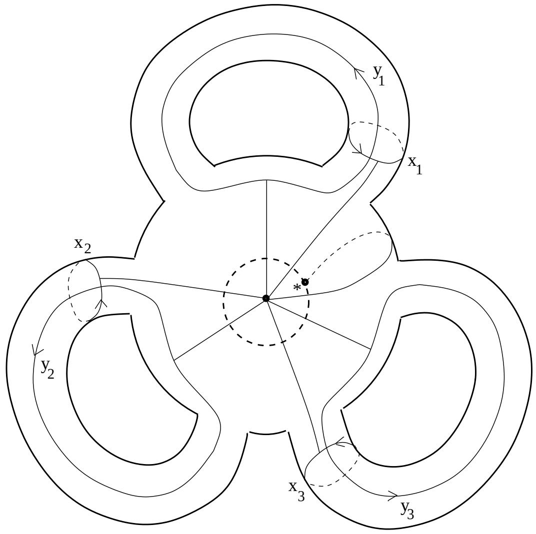

FIG. 3.4 – La base $(x_i, y_i)_{i=1}^{3}$ de $\pi_1(\Sigma_{3,1}, \star)$

et soit $\theta \in \pi_1(\Sigma_{3,1}, \star)$ défini par :

$$\begin{aligned}\theta \quad = \quad & y_3^{-1}x_3y_3x_1x_3^{-1}y_2^{-1}x_2x_1^{-1}y_3^{-1}x_3x_2^{-1}y_1^{-1}x_1y_1x_2x_1^{-1}y_3^{-1}x_3^{-1} \\ & y_3x_1x_2^{-1}y_2x_3x_1^{-1}x_3^{-1}y_2^{-1}x_2y_2x_3x_2^{-1}y_1^{-1}x_1x_3^{-1}y_2^{-1}x_2x_1^{-1}y_3^{-1}x_3y_3x_1.\end{aligned}$$

Alors, l'automorphisme $\pi_1(\Sigma_{3,1}, \star) \xrightarrow[\simeq]{h_*} \pi_1(\Sigma_{3,1}, \star)$, induit par le difféomorphisme Borroméen h, est défini par :

$$\forall w \in \pi_1(\Sigma_{3,1}, \star), \quad h_*(w) = c_\theta \mathsf{H}(w) = \theta \cdot \mathsf{H}(w) \cdot \theta^{-1}.$$

DÉMONSTRATION. On commence par calculer l'endomorphisme :

$$\pi_1(\Sigma_{3,1}, \star') \xrightarrow[\simeq]{h'_*} \pi_1(\Sigma_{3,1}, \star')$$

induit par le difféomorphisme Borroméen h. Pour cela, il suffit de calculer $(\tau_{\alpha_1})'_*$ et $\left(\tau_{\beta_1}^{-1}\right)'_*$ en d'en déduire respectivement par permutation cyclique (123) les twists $(\tau_{\alpha_2})'_*, (\tau_{\alpha_3})'_*$ et $\left(\tau_{\beta_2}^{-1}\right)'_*, \left(\tau_{\beta_3}^{-1}\right)'_*$. En composant ces 6 automorphismes, on trouve alors que l'automorphisme h'_* est donné par H *via* les changements de variables $x_i \leftrightarrow x'_i$ et $y_i \leftrightarrow y'_i$.

Puis, le diagramme suivant commutant :

$$\begin{array}{ccc} \pi_1(\Sigma_{3,1}, \star') & \xrightarrow{h'_*} & \pi_1(\Sigma_{3,1}, \star') \\ {\scriptstyle c_\gamma}\Big\uparrow & & \Big\uparrow{\scriptstyle c_{h(\gamma)}} \\ \pi_1(\Sigma_{3,1}, \star) & \xrightarrow{h_*} & \pi_1(\Sigma_{3,1}, \star) \end{array}$$

on en déduit que :

$$h_* = c_{\overline{\gamma}} \circ \left(c_{\gamma h(\overline{\gamma})} \circ h'_*\right) \circ c_\gamma.$$

Enfin, le lacet $\gamma h(\overline{\gamma}) \in \pi_1(\Sigma_{3,1}, \star')$ est donné par le mot θ *via* les changements de variables $x_i \leftrightarrow x'_i$ et $y_i \leftrightarrow y'_i$. $\square$

3.1.2. *Image du difféomorphisme Borroméen par la représentation de Magnus.* Le difféomorphisme h vivant dans le groupe de Torelli de $\Sigma_{3,1}$, nous pouvons considérer son image par la représentation de Magnus de $\mathcal{T}_{3,1}$ relative aux choix du point $\star$ et de la base $(z_i)_{i=1}^6 = (x_i, y_i)_{i=1}^3$ du groupe libre $\pi_1(\Sigma_{3,1}, \star)$ qui ont été faits dans le paragraphe précédent. Nous obtenons :

$$M(h) = aJ(h_*) \in GL_6(\mathbf{Z}[H])$$

où $H = H_1(\Sigma_{3,1})$ est le groupe abélien librement engendré sur la famille $(z_i)_{i=1}^6$.

Soit aussi un homomorphisme d'anneaux :

$$\mathbf{Z}[H] \xrightarrow{\varphi} \mathbf{F}$$

à valeurs dans un corps commutatif $\mathbf{F}$, tel que :

$$\forall i \in \{1,2,3\},\ \varphi(x_i) = 1.$$

Soit $\mathbf{F}\langle e_{z_1}, \ldots, e_{z_6}\rangle$ le $\mathbf{F}$-espace vectoriel de base $(e_{z_i})_{i=1}^6$ et soit :

$$\mathbf{F}\langle e_{z_1}, \ldots, e_{z_6}\rangle \xrightarrow{h_*} \mathbf{F}\langle e_{z_1}, \ldots, e_{z_6}\rangle$$

l'automorphisme[7] dont la matrice dans la base $(e_{z_i})_{i=1}^6$ est :

$$\varphi M(h) = \varphi aJ(h_*) \in GL_6(\mathbf{F}).$$

Lemme 3.4. *L'automorphisme h_* de $\mathbf{F}\langle e_{z_1}, \ldots, e_{z_6}\rangle$ vérifie pour tout $i \in \{1,2,3\}$:*

(3.1)
$$\begin{aligned}\varphi(y_1y_2y_3)\cdot h_*(e_{x_i}) = \quad & (-1+\varphi(y_{i+1})+\varphi(y_iy_{i+2})+\varphi(y_iy_{i+1})-\varphi(y_1y_2y_3))\cdot e_{x_i}\\ & +\varphi(y_i)\varphi(y_{i+2})(\varphi(y_{i+1})-1)\cdot e_{x_{i+1}}+\\ & \left(-\varphi(y_i)+\varphi(y_i)\varphi(y_{i+2})^{-1}-\varphi(y_iy_{i+1})+\varphi(y_1y_2y_3)\right)\cdot e_{x_{i+2}}\\ & +\varphi(y_i)\left(\varphi(y_{i+2})-1\right)\cdot e_{y_{i+1}}+\varphi(y_i)(1-\varphi(y_{i+1}))\cdot e_{y_{i+2}}.\end{aligned}$$

Démonstration. D'après le Lemme 3.3, l'automorphisme h_* de $\pi_1(\Sigma_{3,1}, \star)$ est la composition $c_\theta \circ \mathsf{H}$, d'où :

$$\varphi M(h) = \varphi aJ(c_\theta)\cdot \varphi aJ(H) \in GL_6(\mathbf{F}).$$

Nous avons :

$$\forall i,j \in \{1,\ldots,6\},\quad \frac{\partial c_\theta(z_j)}{\partial z_i} = \delta_{i,j}\cdot\theta + \left(1-\theta z_j\theta^{-1}\right)\cdot\frac{\partial\theta}{\partial z_i} \in \mathbf{Z}\left[\pi_1(\Sigma_{3,1},\star)\right],$$

et donc :

$$\varphi a\left(\frac{\partial c_\theta(z_j)}{\partial z_i}\right) = \delta_{i,j}\cdot\varphi a(\theta) + (1-\varphi(z_j))\cdot\varphi a\left(\frac{\partial\theta}{\partial z_i}\right) \in \mathbf{F}.$$

De l'expression pour θ donnée par le Lemme 3.3, on déduit que :

$$\varphi a(\theta) = \varphi\left(y_1^{-1}y_2^{-1}y_3^{-1}\right) \in \mathbf{F}.$$

L'endomorphisme de $\mathbf{F}\langle e_{z_1}, \ldots, e_{z_6}\rangle$ défini par la matrice $\varphi aJ(c_\theta)$ est donc donné par les assignations suivantes :

$$(3.2)\qquad \left\{\begin{array}{lcl} e_{x_j} & \longmapsto & \varphi\left(y_1^{-1}y_2^{-1}y_3^{-1}\right)\cdot e_{x_j}\\ e_{y_j} & \longmapsto & \varphi\left(y_1^{-1}y_2^{-1}y_3^{-1}\right)\cdot e_{y_j} + (1-\varphi(y_j))\cdot e_\theta\end{array}\right.$$

$$\text{où}\quad e_\theta = \sum_{i=1}^{6}\varphi a\left(\frac{\partial\theta}{\partial z_i}\right)\cdot e_{z_i}\quad (\text{notation du §2.2.1}).$$

De la formule pour H précisée par le Lemme 3.3, il découle par calcul direct que l'automorphisme de $\mathbf{F}\langle e_{z_1}, \ldots, e_{z_6}\rangle$ défini par la matrice $\varphi aJ(\mathsf{H})$, envoie le vecteur

[7] Pour une interprétation topologique de cet automorphisme, voir la preuve du Lemme 2.2.

de base e_{x_i} sur le terme de droite de l'équation (3.1). Nous déduisons alors de (3.2) que :

$$h_*(e_{x_i}) = \varphi\left(y_1^{-1}y_2^{-1}y_3^{-1}\right) \cdot (\text{ terme de droite de (3.1) })$$
$$+\underbrace{\left[\varphi(y_i)\left(\varphi(y_{i+2})-1\right)\left(1-\varphi(y_{i+1})\right)+\varphi(y_i)\left(1-\varphi(y_{i+1})\right)\left(1-\varphi(y_{i+2})\right)\right]}_{=0}\cdot e_\theta.$$

□

Exemple 3.5. Nous nous intéresserons au cas particulier où, en plus de la condition :

$$\forall i \in \{1,2,3\}, \quad \varphi(x_i) = 1 \in \mathbf{F},$$

déjà requise, nous demandons aussi :

$$\varphi(y_1) = \varphi(y_3) = 1 \in \mathbf{F}.$$

Nous déduisons alors de (3.1) les formules suivantes :

$$\left\{\begin{array}{lcl} h_*(e_{x_1}) & = & e_{x_1} + \left(1-\varphi(y_2)^{-1}\right)\cdot e_{x_2} + \left(\varphi(y_2)^{-1}-1\right)\cdot e_{y_3} \\ h_*(e_{x_2}) & = & e_{x_2} \\ h_*(e_{x_3}) & = & \left(1-2\varphi(y_2)^{-1}+\varphi(y_2)^{-2}\right)\cdot e_{x_2} + \varphi(y_2)^{-1}\cdot e_{x_3} + \left(1-\varphi(y_2)^{-1}\right)\cdot e_{y_1} \end{array}\right.$$

3.2. Une relation multiplicative de degré 1. Comme annoncé, nous étudions maintenant le comportement des torsions abéliennes de Reidemeister-Turaev vis-à-vis des chirurgies le long de clovers bouclés.

Soit M une 3-variété fermée orientée et $G \subset M$ un clover bouclé. Soit aussi un homomorphisme d'anneaux :

$$\mathbf{Z}[H_1(M)] \xrightarrow{\varphi} \mathbf{F}$$

à valeurs dans un corps commutatif, et soit $\alpha \in Spin^c(M)$.
On note (M_G, α_G) la $Spin^c$-variété obtenue de (M,α) par chirurgie le long du clover G et :

$$\mathbf{Z}[H_1(M_G)] \xrightarrow{\varphi_G} \mathbf{F}$$

l'homomorphisme d'anneaux défini par φ et l'isomorphisme d'homologie canonique $H_1(M) \xrightarrow{\simeq} H_1(M_G)$. Nous allons calculer $\tau^{\varphi_G}(M_G,\alpha_G)$ en fonction de $\tau^{\varphi}(M,\alpha)$, à un facteur multiplicatif près dans $\varphi(H_1(M))$.

Simplifions le clover G. Quitte à appliquer le mouvement illustré sur la Fig. 1.4 du Chap. 1, on peut supposer que :

$$l\text{-}deg(G) = 1.$$

Grâce au mouvement dessiné sur la Fig. 1.5 du Chap. 1, nous pouvons aussi supposer que tous les sommets du clover G sont situés sur l'unique boucle de G. Alors, le clover G est une "roue dentée". Soit d le degré de G. Nous orientons la boucle, les feuilles et numérotons les sommets comme indiqué sur la Fig. 3.5. Les classes d'homologie des feuilles et de la boucle orientées sont respectivement notées :

$$f_1,\ldots,f_d \in H_1(M) \quad \text{et} \quad b \in H_1(M).$$

Au côté reliant le sommet i au sommet $i+1$, on associe un entier modulo 2 noté ε_i et égal à 1 ou 0 suivant que le côté est twisté ou pas (relativement aux "dents" portées par ces sommets). Nous posons :

$$\varepsilon = \sum_{i=1}^{d} \varepsilon_i \in \mathbf{Z}_2.$$

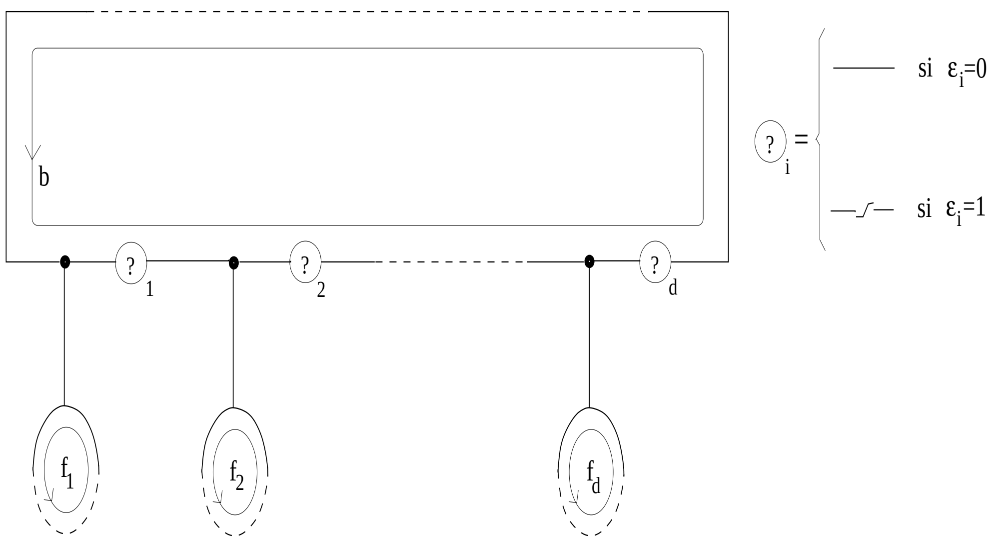

FIG. 3.5 – Le clover bouclé $G \subset M$

LEMME 3.6. *Avec les notations précédentes, nous avons :*

$$\begin{aligned}\tau^{\varphi_G}\left(M_G, \alpha_G\right) \quad = \quad & P_{d,\varepsilon}\left(\varphi(b), \varphi(f_1), \ldots, \varphi(f_d)\right) \cdot \\ & P_{d,\varepsilon}\left(\varphi(b)^{-1}, \varphi(f_1)^{-1}, \ldots, \varphi(f_d)^{-1}\right) \cdot C \cdot \tau^{\varphi}(M, \alpha) \in \mathbf{F},\end{aligned}$$

où $C \in \varphi\left(H_1(M)\right)$ est une constante indépendante de α et où $P_{d,\varepsilon} \in \mathbf{F}[Y, X_1, \ldots, X_d]$ est le polynôme suivant :

$$P_{d,\varepsilon}(Y, X_1, \ldots, X_d) = Y + (-1)^{\varepsilon+1} \prod_{i=1}^{d} (1 - X_i).$$

REMARQUE 3.7. Si les conditions du Lemme 2.14 sur $\mathbf{F}$ et φ sont satisfaites, nous avons alors $C = 1$.

REMARQUE 3.8. Au vu de la Remarque 1.27, il existe un énoncé analogue au Lemme 3.6 pour les polynômes d'Alexander-Conway multivariables des entrelacs orientés et ordonnés des sphères d'homologie orientées (dans ce cas, $C = 1$).
Cet énoncé généralise un résultat de Garoufalidis et Levine relatif aux polynômes d'Alexander des noeuds (voir [**GL**, Lemma 2.5]) et en donne une nouvelle preuve.

DÉMONSTRATION DU LEMME 3.6. Fissionnons le clover G, soit :

$$Y(G) = \left\{G^{(1)}, \ldots, G^{(d)}\right\}$$

la famille de d Y-graphes obtenus. Nous construisons :

$$M_2 = \left(B \cup \bigcup_{i=1}^{d} N\left(G^{(i)}\right)\right) \cup \bigcup_{i=1}^{d} \left(\mathbf{D}^2 \times \mathbf{D}^1\right)^{(i)}$$

où les $N\left(G^{(i)}\right)$ sont des voisinages réguliers deux-à-deux disjoints des graphes $G^{(i)}$, où B est une boule de M disjointe des $N\left(G^{(i)}\right)$ et où $\left(\mathbf{D}^2 \times \mathbf{D}^1\right)^{(i)}$ est une 2-anse telle que $\left(\mathbf{D}^2 \times 0\right)^{(i)} \subset \partial B$ et $\left(\mathbf{D}^2 \times 1\right)^{(i)} \subset \partial N\left(G^{(i)}\right)$. Nous choisissons ces d anses et cette boule B dans une boule voisinage du clover G privé de ses feuilles et du côté reliant le sommet n° d au sommet n° 1. Alors, M_2 est un corps en anses plongé dans M de genre $3d$.

Soit M_1 le complémentaire de M_2. Alors,

$$M = M_1 \cup_f M_2,$$

où $-\partial M_2 \xrightarrow[\simeq]{f} \partial M_1$ est l'identité. Soit aussi $t \in \mathcal{T}(\partial M_2)$ défini par :

$$t(x) = \begin{cases} h^{(i)}(x) & \text{si } x \in \partial N\left(G^{(i)}\right) \setminus \left(\mathbf{D}^2 \times 1\right)^{(i)} \\ x & \text{sinon.} \end{cases}$$

où $h^{(i)}$ est une copie du difféomorphisme Borroméen h pour la surface $\partial N\left(G^{(i)}\right) \setminus \left(\mathbf{D}^2 \times 1\right)^{(i)}$ qui est difféomorphe à $\Sigma_{3,1}$. Alors, (M_G, α_G) est difféomorphe à la $Spin^c$-variété obtenue en twistant (M, α) le long de ∂M_2 par le difféomorphisme t. Aussi, allons-nous pouvoir appliquer les résultats du §2.3, dont nous utiliserons le vocabulaire et les notations.
Nous choisissons un disque D sur $\partial M_2 \cap \partial B$ et un point base $\star$ sur ∂D. Dans la suite, afin de simplifier la notation des groupes d'homologie à coefficients twistés, nous n'indiquons pas la dépendance de ceux-ci en le point $\star$.

Dans le cas "exceptionnel" où pour tout $i \in \{1, \ldots, d\}$, $\varphi(f_i) = 1$, le Lemme 3.6 est une conséquence du Th. 2.12. On se place dans le cas "générique" où il existe un $i_0 \in \{1, \ldots, d\}$ tel que $\varphi(f_{i_0}) \neq 1$. On peut supposer que $i_0 = d$.

Nous notons, comme dans le §2.3, les inclusions :

$$\begin{tikzcd}
& M_1 \arrow[dr, "i_1"] & \\
\partial M_1 \arrow[ur, "j_1"] \arrow[rr, "l_1"] & & M \arrow[d, equal] \\
\partial M_2 \arrow[u, "f = Id"] \arrow[rr, "l_2"] \arrow[dr, "j_2"'] & & M \\
& M_2 \arrow[ur, "i_2"'] &
\end{tikzcd}$$

et, pour $k = 1$ ou 2, nous posons :

$$I_k = Ker\left(H_1^{\varphi l_{k,*}}(\partial M_k) \xrightarrow{j_{k,*}} H_1^{\varphi i_{k,*}}(M_k)\right).$$

Soit $V := H_1^{\varphi l_{1,*}}(\partial M_1) = H_1^{\varphi l_{2,*}}(\partial M_2)$; d'après le Lemme 2.3, V est un $\mathbf{F}$-espace vectoriel de dimension $2 \cdot (3d) - 2 = 6d - 2$. Précisément, nous choisissons une base :

$$\left(x_1^{(i)}, y_1^{(i)}, x_2^{(i)}, y_2^{(i)}, x_3^{(i)}, y_3^{(i)}\right)_{i=1}^{d} = \left(z_1^{(i)}, z_2^{(i)}, z_3^{(i)}, z_4^{(i)}, z_5^{(i)}, z_6^{(i)}\right)_{i=1}^{d}$$

du groupe libre $\pi_1(\partial M_2 \setminus int(D), \star)$. Un cas particulier est dessiné sur la Figure 3.6.
Alors, V s'identifie à :

$$\frac{Ker(m)}{\langle \omega \rangle}.$$

Ici, $Ker(m)$ est l'hyperplan du $\mathbf{F}$-espace-vectoriel :

$$\mathbf{F}\left\langle e_{x_j^{(i)}}, e_{y_j^{(i)}} \ : \ j \in \{1, 2, 3\}, \ i \in \{1, \ldots, d\}\right\rangle.$$

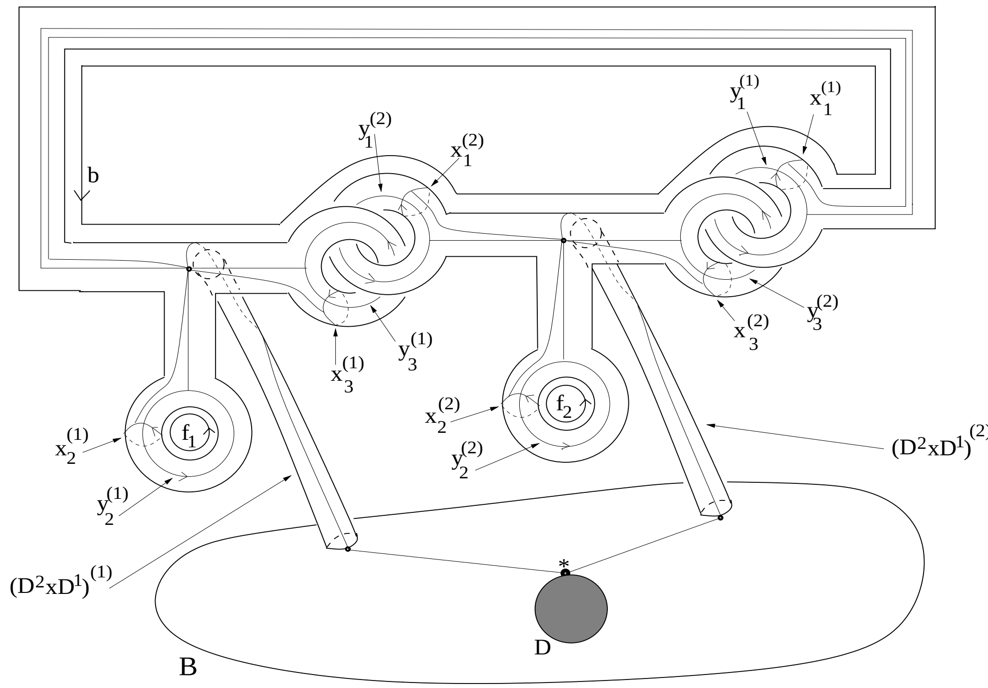

FIG. 3.6 – Un exemple où $d = 2$ et où $\varepsilon_1 = \varepsilon_2 = 0$.

défini par la forme linéaire m telle que :

$$m\left(\sum_{i=1}^{d}\sum_{j=1}^{3}\left(a_j^{(i)}\cdot e_{x_j^{(i)}} + b_j^{(i)}\cdot e_{y_j^{(i)}}\right)\right) = \sum_{i=1}^{d} b_2^{(i)}\left(\varphi(f_i)-1\right) \in \mathbf{F},$$

et ω est le vecteur :

$$\omega = \sum_{i=1}^{d}\left(1-\varphi(f_i)^{-1}\right)\cdot e_{x_2^{(i)}} \in Ker(m).$$

Pour pouvoir évaluer la variation de la torsion, nous nous intéressons aux noyaux I_1 et I_2. Nous affirmons à leur propos les faits suivants :

AFFIRMATION 3.9. Le sous-espace I_2 de V est de dimension $3d-1$ et admet la famille :

$$\left(e_{x_1^{(i)}}\right)_{i=1}^{d} \cup \left(e_{x_3^{(i)}}\right)_{i=1}^{d} \cup \left(e_{x_2^{(i)}}\right)_{i=1}^{d-1}$$

pour base.

AFFIRMATION 3.10. Les $2d$ vecteurs de V qui suivent appartiennent à I_1 (et sont linéairement indépendants) :

$$v_3^{(i)} = \begin{cases} e_{y_3^{(i)}} + (-1)^{\varepsilon_i+1}\cdot e_{x_1^{(i+1)}} & i = 1,\dots,d-1 \\ e_{y_3^{(d)}} + (-1)^{\varepsilon_d+1}\varphi(b)\cdot e_{x_1^{(1)}} & i = d \end{cases}$$

$$v_1^{(i)} = \begin{cases} e_{y_1^{(i)}} + (-1)^{\varepsilon_{i-1}+1}\cdot e_{x_3^{(i-1)}} & i = 2,\dots,d \\ \varphi(b)\cdot e_{y_1^{(1)}} + (-1)^{\varepsilon_d+1}\cdot e_{x_3^{(d)}} & i = 1 \end{cases}$$

Nous déduisons de l'Exemple 3.5 et de la définition des vecteurs $v_1^{(i)}$ et $v_3^{(i)}$ les valeurs suivantes de $I_2 \xrightarrow{t_*|} V$ sur les vecteurs de base de I_2. Pour tout $i \in \{1,\ldots,d-1\}$, nous avons ainsi :

$$\begin{aligned} t_*\left(e_{x_1^{(i)}}\right) &= e_{x_1^{(i)}} + (-1)^{\varepsilon_i}\left(\varphi(f_i)^{-1}-1\right)\cdot e_{x_1^{(i+1)}} \\ &\quad + \left(1-\varphi(f_i)^{-1}\right)\cdot e_{x_2^{(i)}} + \left(\varphi(f_i)^{-1}-1\right)\cdot v_3^{(i)}, \end{aligned}$$

tandis que pour $i=d$:

$$\begin{aligned} t_*\left(e_{x_1^{(d)}}\right) &= e_{x_1^{(d)}} + (-1)^{\varepsilon_d}\varphi(b)\left(\varphi(f_d)^{-1}-1\right)\cdot e_{x_1^{(1)}} \\ &\quad + \left(1-\varphi(f_d)^{-1}\right)\cdot e_{x_2^{(d)}} + \left(\varphi(f_d)^{-1}-1\right)\cdot v_3^{(d)}. \end{aligned}$$

Puis, pour tout $i \in \{2,\ldots,d\}$, nous avons :

$$\begin{aligned} t_*\left(e_{x_3^{(i)}}\right) &= \left(1-\varphi(f_i)^{-1}\right)^2\cdot e_{x_2^{(i)}} + \varphi(f_i)^{-1}\cdot e_{x_3^{(i)}} \\ &\quad +(-1)^{\varepsilon_{i-1}}\left(1-\varphi(f_i)^{-1}\right)\cdot e_{x_3^{(i-1)}} + \left(1-\varphi(f_i)^{-1}\right)\cdot v_1^{(i)}, \end{aligned}$$

tandis que pour $i=1$:

$$\begin{aligned} t_*\left(e_{x_3^{(1)}}\right) &= \left(1-\varphi(f_1)^{-1}\right)^2\cdot e_{x_2^{(1)}} + \varphi(f_1)^{-1}\cdot e_{x_3^{(1)}} + \\ &\quad (-1)^{\varepsilon_d}\varphi(b)^{-1}\left(1-\varphi(f_1)^{-1}\right)\cdot e_{x_3^{(d)}} + \varphi(b)^{-1}\left(1-\varphi(f_1)^{-1}\right)\cdot v_1^{(1)}. \end{aligned}$$

Nous avons enfin pour tout $i \in \{1,\ldots,d-1\}$:

$$t_*\left(e_{x_2^{(i)}}\right) = e_{x_2^{(i)}}.$$

Pour tout $i \in \{1,\ldots,d\}$, notons $q_i = \varphi(f_i)^{-1}-1 \in \mathbf{F}$. Le Th. 2.11 affirme alors que :

$$\tau^{\varphi_G}\left(M_G,\alpha_G\right) = \det(A)\cdot C\cdot \tau^{\varphi}(M,\alpha)$$

où $C \in \varphi\left(H_1(M)\right)$ est une constante indépendante de α et où :

$$A = \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_3 & 0 \\ \hline * & * & Id \end{array}\right) \in \mathcal{M}_{3d-1}(\mathbf{F}).$$

Ici, $A_1 \in \mathcal{M}_d(\mathbf{F})$ est la matrice :

$$A_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & (-1)^{\varepsilon_d}q_d\cdot\varphi(b) \\ (-1)^{\varepsilon_1}q_1 & 1 & \cdots & 0 & 0 \\ 0 & (-1)^{\varepsilon_2}q_2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & (-1)^{\varepsilon_{d-1}}q_{d-1} & 1 \end{pmatrix},$$

et $A_3 \in \mathcal{M}_d(\mathbf{F})$ est la matrice :

$$A_3 = \begin{pmatrix} \varphi(f_1)^{-1} & (-1)^{\varepsilon_1+1}q_2 & 0 & \cdots & 0 \\ 0 & \varphi(f_2)^{-1} & (-1)^{\varepsilon_2+1}q_3 & \cdots & 0 \\ 0 & 0 & \varphi(f_3)^{-1} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & (-1)^{\varepsilon_{d-1}+1}q_d \\ (-1)^{\varepsilon_d+1}\varphi(b)^{-1}\cdot q_1 & 0 & 0 & \cdots & \varphi(f_d)^{-1} \end{pmatrix}.$$

En développant $\det(A_1)$ par rapport à la première ligne et $\det(A_3)$ par rapport à la première colonne, nous obtenons :

$$\det(A) = \left(1 + (-1)^{d+1}\varphi(b)\prod_{i=1}^{d}(-1)^{\varepsilon_i}q_i\right)\cdot$$
$$\left(\prod_{i=1}^{d}\varphi(f_i)^{-1} + (-1)^{d+1}\varphi(b)^{-1}\prod_{i=1}^{d}(-1)^{\varepsilon_i+1}q_{i+1}\right),$$

d'où nous déduisons que :

$$\det(A) = P_{d,\varepsilon}\left(\varphi(b),\varphi(f_1),\dots,\varphi(f_d)\right)\cdot P_{d,\varepsilon}\left(\varphi(b)^{-1},\varphi(f_1)^{-1},\dots,\varphi(f_d)^{-1}\right)$$

à une constante mutiplicative près dans $\varphi\left(H_1(M)\right)$.
Nous prouvons maintenant les affirmations qui restent en suspens.

Démonstration de l'Aff. 3.9. Puisque $x_j^{(i)} = 1 \in \pi_1\left(M_2,\star\right)$ pour tout $i \in \{1,\dots,d\}$ et pour tout $j \in \{1,2,3\}$, il découle du Lemme 2.2 d) que les vecteurs $e_{x_j^{(i)}}$ appartiennent à I_2. L'espace V n'étant que le quotient de l'hyperplan $Ker(m)$ par la droite engendrée par ω, il s'en suit l'indépendance linéaire des $3d-1$ vecteurs $e_{x_j^{(i)}}$ pour $(i,j) \in \{1,\dots,d\}\times\{1,2,3\}\setminus\{(d,2)\}$. Enfin, le corps en anses M_2 se rétractant à un bouquet de $3d$ cercles, on calcule facilement le groupe d'homologie à coefficients twistés $H_1^{\varphi i_{2,*}}(M_2)$ (comme dans la preuve du Lemme 2.2) : c'est un $\mathbf{F}$-espace vectoriel de dimension $3d-1$ qui est isomorphe à $\left(Ker(m)\cap\left\langle e_{y_j^{(i)}} : j=1,2,3,\ i=1,\dots,d\right\rangle\right)/\langle\omega\rangle$ par $j_{2,*}$. Il vient en particulier que $j_{2,*}$ est surjective, d'où :

$$\dim(I_2) = \dim(V) - \dim\left(H_1^{\varphi i_{2,*}}(M_2)\right) = (6d-2)-(3d-1) = 3d-1.$$

□

Démonstration de l'Aff. 3.10. Pour tout $i \in \{1,\dots,d\}$, il existe des lacets $w_1^{(i)} \in \pi_1\left(\partial M_2,\star\right)$ et $w_3^{(i)} \in \pi_1\left(\partial M_2,\star\right)$ tels que :

$$\left\{\begin{array}{l} x_3^{(i)} = w_1^{(i+1)}\cdot\left(y_1^{(i+1)}\right)^{(-1)^{\varepsilon_i}}\cdot\left(w_1^{(i+1)}\right)^{-1} \in \pi_1\left(M_1,\star\right) \\ x_1^{(i+1)} = w_3^{(i)}\cdot\left(y_3^{(i)}\right)^{(-1)^{\varepsilon_i}}\cdot\left(w_3^{(i)}\right)^{-1} \in \pi_1\left(M_1,\star\right) \end{array}\right.$$

Nous allons appliquer le Lemme 2.2 d) ; nous calculons pour cela les vecteurs :

$$e_{\left(x_3^{(i)}\right)^{-1}\cdot w_1^{(i+1)}\cdot\left(y_1^{(i+1)}\right)^{(-1)^{\varepsilon_i}}\cdot\left(w_1^{(i+1)}\right)^{-1}} \quad \text{et :} \quad e_{\left(x_1^{(i+1)}\right)^{-1}\cdot w_3^{(i)}\cdot\left(y_3^{(i)}\right)^{(-1)^{\varepsilon_i}}\cdot\left(w_3^{(i)}\right)^{-1}}$$

de l'espace vectoriel :

$$\mathbf{F}\left\langle e_{z_j^{(i)}}\ :\ j\in\{1,\dots,6\},\ i\in\{1,\dots,d\}\right\rangle.$$

Puisque $\varphi\left(x_j^{(i)}\right) = \varphi\left(y_j^{(i)}\right) = 1$ pour tout $i = 1,\dots,d$ et pour $j = 1,3$, nous obtenons pour tous $k \in \{1,\dots,6\}$ et $l \in \{1,\dots,d\}$:

$$\varphi a\left(\frac{\partial}{\partial z_k^{(l)}}\left(\left(x_3^{(i)}\right)^{-1} w_1^{(i+1)}\left(y_1^{(i+1)}\right)^{(-1)^{\varepsilon_i}}\left(w_1^{(i+1)}\right)^{-1}\right)\right) =$$
$$-\varphi a\left(\frac{\partial x_3^{(i)}}{\partial z_k^{(l)}}\right) + (-1)^{\varepsilon_i}\varphi\left(w_1^{(i+1)}\right)\varphi a\left(\frac{\partial y_1^{(i+1)}}{\partial z_k^{(l)}}\right),$$
$$\text{et :}\quad \varphi a\left(\frac{\partial}{\partial z_k^{(l)}}\left(\left(x_1^{(i+1)}\right)^{-1} w_3^{(i)}\left(y_3^{(i)}\right)^{(-1)^{\varepsilon_i}}\left(w_3^{(i)}\right)^{-1}\right)\right) =$$

$$-\varphi a\left(\frac{\partial x_1^{(i+1)}}{\partial z_k^{(l)}}\right)+(-1)^{\varepsilon_i}\varphi\left(w_3^{(i)}\right)\varphi a\left(\frac{\partial y_3^{(i)}}{\partial z_k^{(l)}}\right).$$

si bien que :

$$\begin{aligned}
e_{\left(x_3^{(i)}\right)^{-1}\cdot w_1^{(i+1)}\cdot\left(y_1^{(i+1)}\right)^{(-1)^{\varepsilon_i}}\cdot\left(w_1^{(i+1)}\right)^{-1}} &= -e_{x_3^{(i)}}+(-1)^{\epsilon_i}\varphi\left(w_1^{(i+1)}\right)\cdot e_{y_1^{(i+1)}},\\
e_{\left(x_1^{(i+1)}\right)^{-1}\cdot w_3^{(i)}\cdot\left(y_3^{(i)}\right)^{(-1)^{\varepsilon_i}}\cdot\left(w_3^{(i)}\right)^{-1}} &= -e_{x_1^{(i+1)}}+(-1)^{\varepsilon_i}\varphi\left(w_3^{(i)}\right)\cdot e_{y_3^{(i)}}.
\end{aligned}$$

Nous rappelant la façon particulière dont nous avons choisi les 2-anses, nous voyons que :

$$\begin{aligned}
\varphi\left(w_1^{(i+1)}\right) &= \begin{cases} 1 & i\in\{1,\ldots,d-1\}\\ \varphi(b) & i=d\end{cases}\\
\varphi\left(w_3^{(i)}\right) &= \begin{cases} 1 & i\in\{1,\ldots,d-1\}\\ \varphi(b)^{-1} & i=d\end{cases}
\end{aligned}$$

Nous en déduisons que :

$$\begin{aligned}
e_{\left(x_3^{(i)}\right)^{-1}\cdot w_1^{(i+1)}\cdot\left(y_1^{(i+1)}\right)^{(-1)^{\varepsilon_i}}\cdot\left(w_1^{(i+1)}\right)^{-1}} &= (-1)^{\epsilon_i}\cdot v_1^{(i+1)}\\
e_{\left(x_1^{(i+1)}\right)^{-1}\cdot w_3^{(i)}\cdot\left(y_3^{(i)}\right)^{(-1)^{\varepsilon_i}}\cdot\left(w_3^{(i)}\right)^{-1}} &= (-1)^{\epsilon_i}\varphi\left(w_3^{(i)}\right)\cdot v_3^{(i)},
\end{aligned}$$

puis concluons alors grâce au Lemme 2.3 d). □

Cela achève la preuve du Lemme 3.6. □

THÉORÈME 3.11. *Soient M une 3-variété fermée orientée, $\mathbf{Z}[H_1(M)]\xrightarrow{\varphi}\mathbf{F}$ un homomorphisme d'anneaux à valeurs dans un corps commutatif, et G_1, G_2 des clovers de M disjoints tels que :*

$$l\text{-}deg\,(G_1)+l\text{-}deg\,(G_2)\geq 1.$$

Alors, nous avons pour toute $\alpha\in Spin^c(M)$:

$$\tau^{\varphi_{G_1,G_2}}\left(M_{G_1,G_2},\alpha_{G_1,G_2}\right)\cdot\tau^{\varphi}\left(M,\alpha\right)=C\cdot\tau^{\varphi_{G_1}}\left(M_{G_1},\alpha_{G_1}\right)\cdot\tau^{\varphi_{G_2}}\left(M_{G_2},\alpha_{G_2}\right)\in\mathbf{F},$$

où $C\in\varphi\left(H_1(M)\right)$ est une constante indépendante de α.

REMARQUE 3.12. Si les conditions du Lemme 2.14 sur $\mathbf{F}$ et φ sont satisfaites, nous avons alors $C=1$.

DÉMONSTRATION DU THÉORÈME 3.11. Supposons que $l\text{-}deg\,(G_1)\geq 1$. Comme il a été décrit avant le Lemme 3.6, on peut supposer que le clover G_1 est une “roue dentée”. Alors, ce lemme-là appliqué à $G_1\subset M$ et à $G_1\subset M_{G_2}$ donne respectivement :

$$\begin{aligned}
\tau^{\varphi_{G_1}}\left(M_{G_1},\alpha_{G_1}\right) &= C_{G_1\subset M}\cdot Q_{G_1\subset M}\cdot\tau^{\varphi}\left(M,\alpha\right),\\
\tau^{\left(\varphi_{G_2}\right)_{G_1}}\left(\left(M_{G_2}\right)_{G_1},\left(\alpha_{G_2}\right)_{G_1}\right) &= C_{G_1\subset M_{G_2}}\cdot Q_{G_1\subset M_{G_2}}\cdot\tau^{\varphi_{G_2}}\left(M_{G_2},\alpha_{G_2}\right).
\end{aligned}$$

Ici, $C_{G_1\subset M}$ et $C_{G_1\subset M_{G_2}}$ appartiennent à $\varphi\left(H_1(M)\right)=\varphi_{G_2}\left(H_1\left(M_{G_2}\right)\right)$ et ne dépendent pas de α, tandis que $Q_{G_1\subset M}\in\mathbf{F}$ (resp. $Q_{G_1\subset M_{G_2}}\in\mathbf{F}$) ne dépend que des valeurs de φ (resp. de φ_{G_2}) sur les classes d'homologies définies par le clover G_1 dans M (resp. dans M_{G_2}). Le diagramme suivant étant commutatif :

$$\begin{tikzcd}
 & H_1(M) \arrow[dd, "\simeq"'] \arrow[dr, "\varphi"] & \\
H_1\left(M \setminus N(G_2)\right) \arrow[ur] \arrow[dr] & & \mathbf{F} \\
 & H_1\left(M_{G_2}\right) \arrow[ur, "\varphi_{G_2}"'] &
\end{tikzcd}$$

nous avons : $Q_{G_1 \subset M} = Q_{G_1 \subset M_{G_2}}$, ce qui nous permet de conclure. □

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